The Kinematic Equations

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Introduction

The *kinematic* or *strain-displacement* equations describe how the strains – the stretching and distortion – within a loaded body relate to the body's displacements. The displacement components in the x, y, and z directions are denoted by the vector $\mathbf{u} \equiv u_i \equiv (u, v, w)$, and are functions of position within the body: $\mathbf{u} = \mathbf{u}(x, y, z)$. If all points within the material experience the same displacement ($\mathbf{u} =$ constant), the structure moves as a rigid body, but does not stretch or deform internally. For stretching to occur, points within the body must experience *different* displacements.

Infinitesimal strain

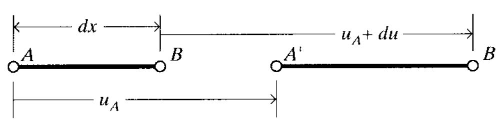

Figure 1: Incremental deformation.

Consider two points A and B separated initially by a small distance dx as shown in Fig. 1, and experiencing motion in the x direction. If the displacement at point A is u_A, the displacement at B can be expressed by a Taylor's series expansion of $u(x)$ around the point $x = A$:

$$u_B = u_A + du = u_A + \frac{\partial u}{\partial x} dx$$

where here the expansion has been truncated after the second term. The *differential* motion δ between the two points is then

$$\delta = u_B - u_A = \left(u_A + \frac{\partial u}{\partial x} dx \right) - u_A = \frac{\partial u}{\partial x} dx$$

In our concept of stretching as being the differential displacement per unit length, the x component of strain is then

$$\boxed{\epsilon_x = \frac{\delta}{dx} = \frac{\partial u}{\partial x}} \tag{1}$$

Hence the strain is a *displacement gradient.* Applying similar reasoning to differential motion in the y direction, the y-component of strain is the gradient of the vertical displacement v with respect to y:

$$\boxed{\epsilon_y = \frac{\partial v}{\partial y}} \tag{2}$$

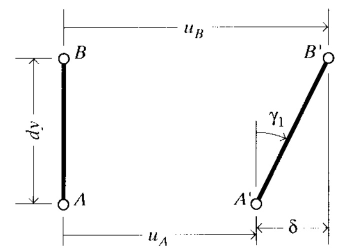

Figure 2: Shearing distortion.

The distortion of the material, which can be described as the change in originally right angles, is the sum of the tilts imparted to vertical and horizontal lines. As shown in Fig. 2, the tilt of an originally vertical line is the relative horizontal displacement of two nearby points along the line:

$$\delta = u_B - u_A = \left(u_A + \frac{\partial u}{\partial y}dy\right) - u_A = \frac{\partial u}{\partial y}dy$$

The change in angle is then

$$\gamma_1 \approx \tan\gamma_1 = \frac{\delta}{dy} = \frac{\partial u}{\partial y}$$

Similarly (see Fig. 3), the tilt γ_2 of an originally horizontal line is the gradient of v with respect to x. The shear strain in the xy plane is then

$$\boxed{\gamma_{xy} = \gamma_1 + \gamma_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}} \tag{3}$$

This notation, using ϵ for normal strain and γ for shearing strain, is sometimes known as the "classical" description of strain.

Matrix Formulation

The "indicial notation" described in the Module on Matrix and Index Notation provides a concise method of writing out all the components of three-dimensional states of strain:

$$\boxed{\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)} \tag{4}$$

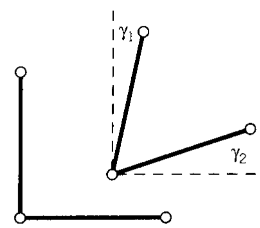

Figure 3: Shearing strain.

where the comma denotes differentiation with respect to the following spatial variable. This double-subscript index notation leads naturally to a matrix arrangement of the strain components, in which the *i*-*j* component of the strain becomes the matrix element in the i^{th} row and the j^{th} column:

$$\epsilon_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \tag{5}$$

Note that the strain matrix is symmetric, i.e. $\epsilon_{ij} = \epsilon_{ji}$. This symmetry means that there are six rather than nine independent strains, as might be expected in a 3×3 matrix. Also note that the indicial description of strain yields the same result for the normal components as in the classical description: $\epsilon_{11} = \epsilon_x$. However, the indicial components of shear strain are half their classical counterparts: $\epsilon_{12} = \gamma_{xy}/2$.

In still another useful notational scheme, the classical strain-displacement equations can be written out in a vertical list, similar to a vector:

$$\left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \\ \partial v/\partial z + \partial w/\partial y \\ \partial u/\partial z + \partial w/\partial x \\ \partial u/\partial y + \partial v/\partial x \end{array} \right\}$$

This vector-like arrangement of the strain components is for convenience only, and is sometimes called a *pseudovector*. Strain is actually a *second-rank tensor,* like stress or moment of inertia, and has mathematical properties very different than those of vectors. The ordering of the elements in the pseudovector form is arbitrary, but it is conventional to list them as we have here by moving down the diagonal of the strain matrix of Eqn. 5 from upper left to lower right, then move up the third column, and finally move one column to the left on the first row; this gives the ordering 1,1; 2,2; 3,3; 2,3; 1,3; 1,2.

Following the rules of matrix multiplication, the strain pseudovector can also be written in terms of the displacement vector as

$$\left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{array} \right\} = \left[\begin{array}{ccc} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \\ \partial/\partial y & \partial/\partial x & 0 \end{array} \right] \left\{ \begin{array}{c} u \\ v \\ w \end{array} \right\} \tag{6}$$

The matrix in brackets above, whose elements are differential operators, can be abbreviated as **L**:

$$\mathbf{L} = \left[\begin{array}{ccc} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial y \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \\ \partial/\partial y & \partial/\partial x & 0 \end{array} \right] \tag{7}$$

The strain-displacement equations can then be written in the concise "pseudovector-matrix" form:

$$\boxed{\boldsymbol{\epsilon} = \mathbf{L}\mathbf{u}} \tag{8}$$

Equations such as this must be used in a well-defined context, as they apply only when the somewhat arbitrary pseudovector listing of the strain components is used.

Volumetric strain

Since the normal strain is just the change in length per unit of original length, the new length L' after straining is found as

$$\epsilon = \frac{L' - L_0}{L_0} \Rightarrow L' = (1+\epsilon)L_0 \tag{9}$$

If a cubical volume element, originally of dimension *abc*, is subjected to normal strains in all three directions, the change in the element's volume is

$$\frac{\Delta V}{V} = \frac{a'b'c' - abc}{abc} = \frac{a(1+\epsilon_x)\,b(1+\epsilon_y)\,c(1+\epsilon_z) - abc}{abc}$$

$$= (1+\epsilon_x)\,(1+\epsilon_y)\,(1+\epsilon_z) - 1 \approx \epsilon_x + \epsilon_y + \epsilon_z \tag{10}$$

where products of strains are neglected in comparison with individual values. The volumetric strain is therefore the sum of the normal strains, i.e. the sum of the diagonal elements in the strain matrix (this is also called the *trace* of the matrix, or $\mathrm{Tr}[\boldsymbol{\epsilon}]$). In index notation, this can be written simply

$$\frac{\Delta V}{V} = \epsilon_{kk}$$

This is known as the volumetric, or "dilatational" component of the strain.

Example 1

To illustrate how volumetric strain is calculated, consider a thin sheet of steel subjected to strains in its plane given by $\epsilon_x = 3$, $\epsilon_y = -4$, and $\gamma_{xy} = 6$ (all in μin/in). The sheet is not in plane strain, since it can undergo a Poisson strain in the z direction given by $\epsilon_z = -\nu(\epsilon_x + \epsilon_y) = -0.3(3-4) = 0.3$. The total state of strain can therefore be written as the matrix

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} 3 & 6 & 0 \\ 6 & -4 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \times 10^{-6}$$

where the brackets on the $[\boldsymbol{\epsilon}]$ symbol emphasize that the matrix rather than pseudovector form of the strain is being used. The volumetric strain is:

$$\frac{\Delta V}{V} = (3 - 4 + 0.3) \times 10^{-6} = -0.7 \times 10^{-6}$$

Engineers often refer to "microinches" of strain; they really mean microinches per inch. In the case of volumetric strain, the corresponding (but awkward) unit would be micro-cubic-inches per cubic inch.

Finite strain

The infinitesimal strain-displacement relations given by Eqns. 3.1–3.3 are used in the vast majority of mechanical analyses, but they do not describe stretching accurately when the displacement gradients become large. This often occurs when polymers (especially elastomers) are being considered. Large strains also occur during deformation processing operations, such as stamping of steel automotive body panels. The kinematics of large displacement or strain can be complicated and subtle, but the following section will outline a simple description of *Lagrangian* finite strain to illustrate some of the concepts involved.

Consider two orthogonal lines OB and OA as shown in Fig. 4, originally of length dx and dy, along the x-y axes, where for convenience we set $dx = dy = 1$. After strain, the endpoints of these lines move to new positions $A_1O_1B_1$ as shown. We will describe these new positions using the coordinate scheme of the original x-y axes, although we could also allow the new positions to define a new set of axes. In following the motion of the lines with respect to the original positions, we are using the so-called *Lagrangian* viewpoint. We could alternately have used the final positions as our reference; this is the *Eulerian* view often used in fluid mechanics.

After straining, the distance dx becomes

$$(dx)' = \left(1 + \frac{\partial u}{\partial x}\right) dx$$

Using our earlier "small" thinking, the x-direction strain would be just $\partial u/\partial x$. But when the strains become larger, we must also consider that the upward motion of point B_1 relative to O_1, that is $\partial v/\partial x$, also helps stretch the line OB. Considering both these effects, the Pythagorean theorem gives the new length O_1B_1 as

$$O_1B_1 = \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

We now define our Lagrangian strain as

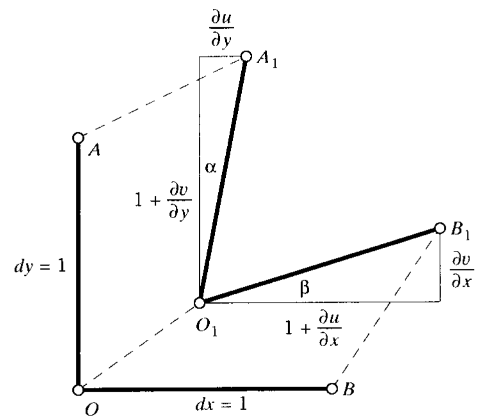

Figure 4: Finite displacements.

$$\epsilon_x = \frac{O_1B_1 - OB}{OB} = O_1B_1 - 1$$

$$= \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} - 1$$

Using the series expansion $\sqrt{1+x} = 1 + x/2 + x^2/8 + \cdots$ and neglecting terms beyond first order, this becomes

$$\epsilon_x \approx \left\{1 + \frac{1}{2}\left[2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\right\} - 1$$

$$= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \tag{11}$$

Similarly, we can show

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \tag{12}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} \tag{13}$$

When the strains are sufficiently small that the quadratic terms are negligible compared with the linear ones, these reduce to the infinitesimal-strain expressions shown earlier.

Example 2

The displacement function $u(x)$ for a tensile specimen of uniform cross section and length L, fixed at one end and subjected to a displacement δ at the other, is just the linear relation

$$u(x) = \left(\frac{x}{L}\right)\delta$$

The Lagrangian strain is then given by Eqn. 11 as

$$\epsilon_x = \frac{\delta}{L} + \frac{1}{2}\left(\frac{\delta}{L}\right)^2$$

The first term is the familiar small-strain expression, with the second nonlinear term becoming more important as δ becomes larger. When $\delta = L$, i.e. the conventional strain is 100%, there is a 50% difference between the conventional and Lagrangian strain measures.

The Lagrangian strain components can be generalized using index notation as

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{r,i}\, u_{r,j}).$$

A pseudovector form is also convenient occasionally:

$$\left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array} \right\} = \left\{ \begin{array}{c} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{array} \right\} + \frac{1}{2} \left[\begin{array}{cccc} u_{,x} & v_{,x} & 0 & 0 \\ 0 & 0 & u_{,y} & v_{,y} \\ u_{,y} & v_{,y} & u_{,x} & v_{,x} \end{array} \right] \left\{ \begin{array}{c} u_{,x} \\ v_{,x} \\ u_{,y} \\ v_{,y} \end{array} \right\}$$

$$= \left(\left[\begin{array}{cc} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{array} \right] + \frac{1}{2} \left[\begin{array}{cccc} u_{,x} & v_{,x} & 0 & 0 \\ 0 & 0 & u_{,y} & v_{,y} \\ u_{,y} & v_{,y} & u_{,x} & v_{,x} \end{array} \right] \left[\begin{array}{cc} \partial/\partial x & 0 \\ 0 & \partial/\partial x \\ \partial/\partial y & 0 \\ 0 & \partial/\partial y \end{array} \right] \right) \left\{ \begin{array}{c} u \\ v \end{array} \right\}$$

which can be abbreviated

$$\boldsymbol{\epsilon} = [\mathbf{L} + \mathbf{A}(\mathbf{u})]\, \mathbf{u} \tag{14}$$

The matrix $\mathbf{A}(\mathbf{u})$ contains the nonlinear effect of large strain, and becomes negligible when strains are small.

Problems

1. Write out the abbreviated strain-displacement equation $\boldsymbol{\epsilon} = \mathbf{L}\mathbf{u}$ (Eqn. 8) for two dimensions.

2. Write out the components of the Lagrangian strain tensor in three dimensions:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{r,i}\, u_{r,j})$$

3. Show that for small strains the fractional volume change is the trace of the infinitesimal strain tensor:

$$\frac{\Delta V}{V} \equiv \epsilon_{kk} = \epsilon_x + \epsilon_y + \epsilon_z$$

4. When the material is incompressible, show the extension ratios are related by

$$\lambda_x \lambda_y \lambda_z = 1$$

5. Show that the kinematic (strain-displacement) relations in for polar coordinates can be written

$$\epsilon_r = \frac{\partial u_r}{\partial r}$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$