UNDERSTANDING PROGRAM EFFICIENCY: 2

(download slides and .py files and follow along!)

6.0001 LECTURE 11

TODAY

- Classes of complexity
- Examples characteristic of each class

WHY WE WANT TO UNDERSTAND EFFICIENCY OF PROGRAMS

- how can we reason about an algorithm in order to predict the amount of time it will need to solve a problem of a particular size?
- how can we relate choices in algorithm design to the time efficiency of the resulting algorithm?
    - are there fundamental limits on the amount of time we will need to solve a particular problem?

ORDERS OF GROWTH: RECAP

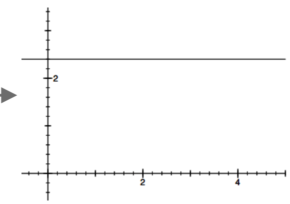
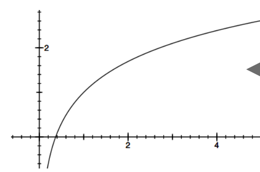
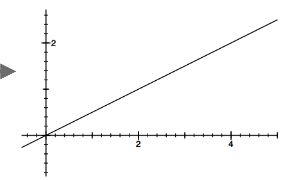
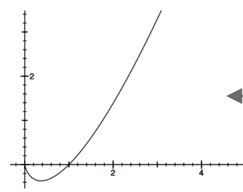
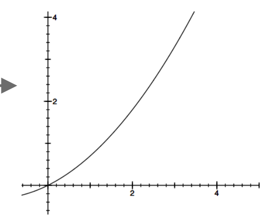
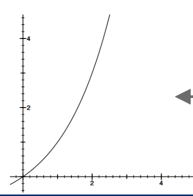
Goals:

- want to evaluate program's efficiency when **input is very big**
- want to express the **growth of program's run time** as input size grows
- want to put an **upper bound** on growth – as tight as possible
- do not need to be precise: **"order of" not "exact"** growth
- we will look at **largest factors** in run time (which section of the program will take the longest to run?)
- **thus, generally we want tight upper bound on growth, as function of size of input, in worst case**

COMPLEXITY CLASSES: RECAP

- $O(1)$ denotes constant running time
- $O(\log n)$ denotes logarithmic running time
- $O(n)$ denotes linear running time
- $O(n \log n)$ denotes log-linear running time
- $O(n^c)$ denotes polynomial running time (c is a constant)
- $O(c^n)$ denotes exponential running time (c is a constant being raised to a power based on size of input)

COMPLEXITY CLASSES ORDERED LOW TO HIGH

O(1)	:	constant	
O(log n)	:		logarithmic
O(n)	:	linear	
O(n log n)	:		loglinear
$O(n^c)$	:	polynomial	
$O(c^n)$	:		exponential

c is a constant

COMPLEXITY GROWTH

CLASS	n=10	= 100	= 1000	= 1000000
O(1)	1	1	1	1
O(log n)	1	2	3	6
O(n)	10	100	1000	1000000
O(n log n)	10	200	3000	6000000
O(n^2)	100	10000	1000000	1000000000000
O(2^n)	1024	12676506 00228229 40149670 3205376	1071508607186267320948425049060 0018105614048117055336074437503 8837035105112493612249319837881 5695858127594672917553146825187 1452856923140435984577574698574 8039345677748242309854210746050 6237114187795418215304647498358 1941267398767559165543946077062 9145711964776865421676604298316 52624386837205668069376	**Good luck!!**

CONSTANT COMPLEXITY

- complexity independent of inputs
- very few interesting algorithms in this class, but can often have pieces that fit this class
- can have loops or recursive calls, but ONLY IF number of iterations or calls independent of size of input

LOGARITHMIC COMPLEXITY

- complexity grows as log of size of one of its inputs
- example:
  - bisection search
  - binary search of a list

BISECTION SEARCH

- suppose we want to know if a particular element is present in a list
- saw last time that we could just “walk down” the list, checking each element
- complexity was linear in length of the list
- suppose we know that the list is ordered from smallest to largest
  - saw that sequential search was still linear in complexity
  - can we do better?

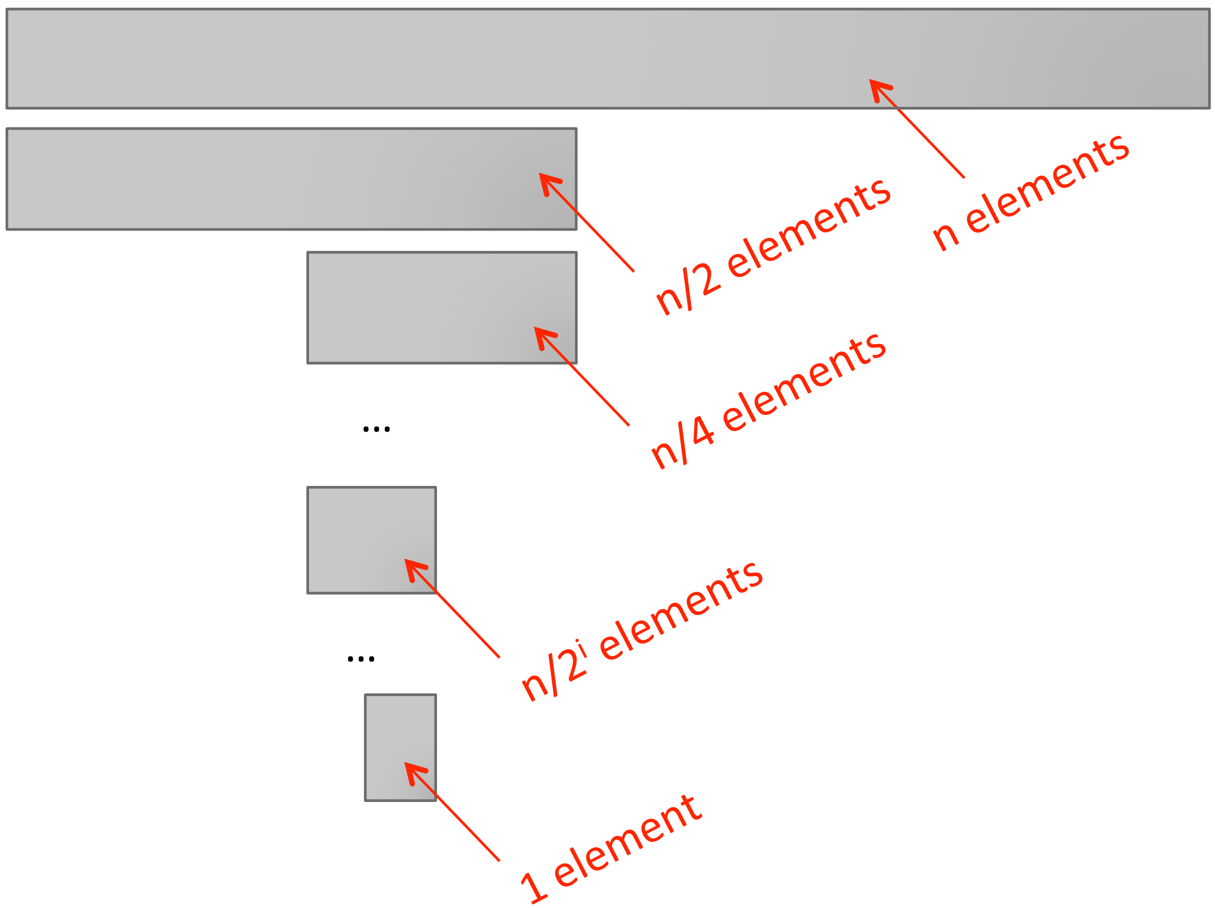
BISECTION SEARCH

1. pick an index, `i`, that divides list in half
2. ask if `L[i] == e`
3. if not, ask if `L[i]` is larger or smaller than `e`
4. depending on answer, search left or right half of `L` for `e`

A new version of a divide-and-conquer algorithm

- break into smaller version of problem (smaller list), plus some simple operations
- answer to smaller version is answer to original problem

BISECTION SEARCH COMPLEXITY ANALYSIS



- finish looking through list when

  $1 = n/2^i$

  so $i = \log n$

- complexity of recursion is **O(log n) – where n is len(L)**

BISECTION SEARCH IMPLEMENTATION 1

```
def bisect_search1(L, e):
    if L == []:
        return False
    elif len(L) == 1:
        return L[0] == e
    else:
        half = len(L)//2
        if L[half] > e:
            return bisect_search1( L[:half], e)
        else:
            return bisect_search1( L[half:], e)
```

constant O(1)

constant O(1)

constant O(1)

NOT constant, copies list

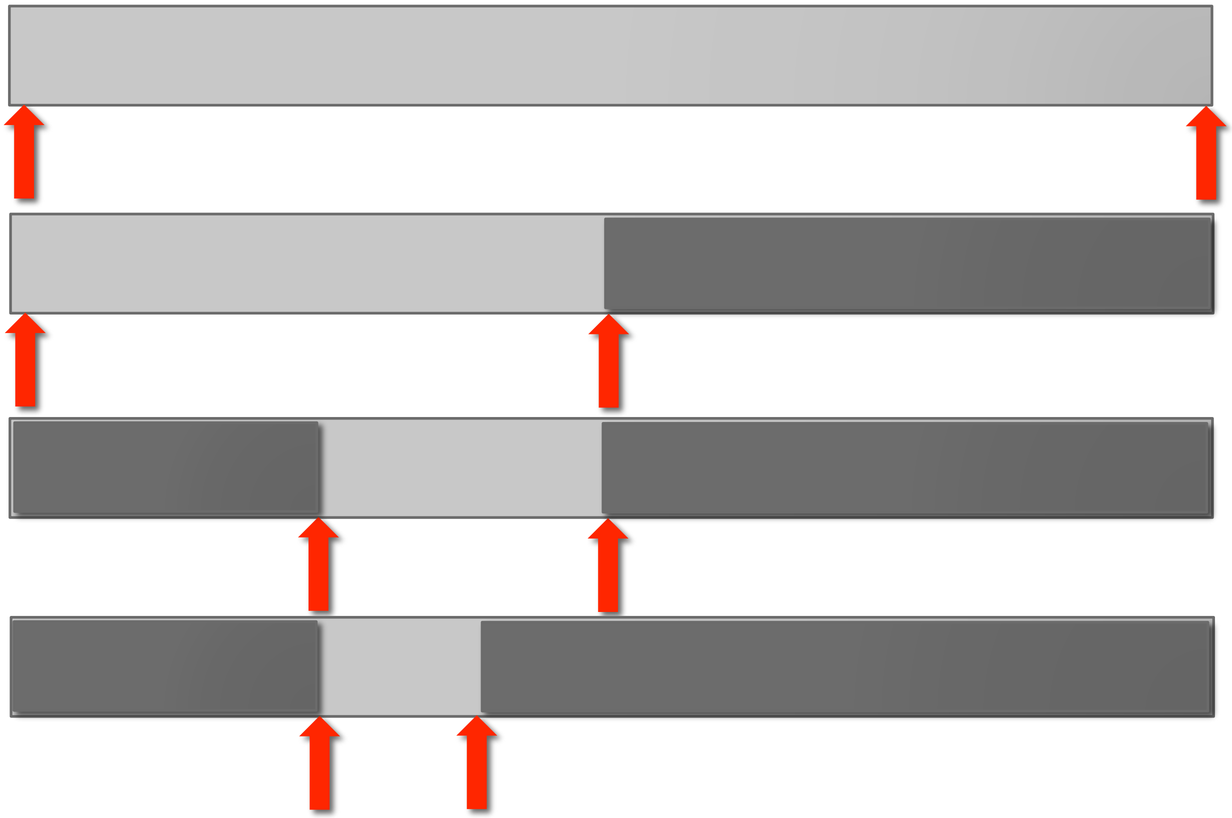
NOT constant

NOT constant

COMPLEXITY OF FIRST BISECTION SEARCH METHOD

- **implementation 1 – bisect_search1**
  - O(log n) bisection search calls
    - On each recursive call, size of range to be searched is cut in half
    - If original range is of size n, in worst case down to range of size 1 when n/(2^k) = 1; or when k = log n
  - O(n) for each bisection search call to copy list
    - This is the cost to set up each call, so do this for each level of recursion
  - O(log n) * O(n) → **O(n log n)**
  - if we are really careful, note that length of list to be copied is also halved on each recursive call
    - turns out that total cost to copy is **O(n)** and this dominates the log n cost due to the recursive calls

BISECTION SEARCH ALTERNATIVE



- still reduce size of problem by factor of two on each step
- but just keep track of low and high portion of list to be searched
- avoid copying the list
- complexity of recursion is again **O(log n) – where n is len(L)**

BISECTION SEARCH IMPLEMENTATION 2

```
def bisect_search2(L, e):
    def bisect_search_helper(L, e, low, high):
        if high == low:
            return L[low] == e
        mid = (low + high)//2
        if L[mid] == e:
            return True
        elif L[mid] > e:
            if low == mid: #nothing left to search
                return False
            else:
                return bisect_search_helper(L, e, low, mid - 1)
        else:
            return bisect_search_helper(L, e, mid + 1, high)
    if len(L) == 0:
        return False
    else:
        return bisect_search_helper(L, e, 0, len(L) - 1)
```

constant other than recursive call

constant other than recursive call

COMPLEXITY OF SECOND BISECTION SEARCH METHOD

- **implementation 2 – bisect_search2** and its helper
  - O(log n) bisection search calls
    - On each recursive call, size of range to be searched is cut in half
    - If original range is of size n, in worst case down to range of size 1 when n/(2^k) = 1; or when k = log n
  - pass list and indices as parameters
  - list never copied, just re-passed as a pointer
  - thus O(1) work on each recursive call
  - O(log n) * O(1) → **O(log n)**

LOGARITHMIC COMPLEXITY

```python
def intToStr(i):
    digits = '0123456789'
    if i == 0:
        return '0'
    result = ''
    while i > 0:
        result = digits[i%10] + result
        i = i//10
    return result
```

LOGARITHMIC COMPLEXITY

```
def intToStr(i):
    digits = '0123456789'
    if i == 0:
        return '0'
    res = ''
    while i > 0:
        res = digits[i%10] + res
        i = i//10
    return result
```

only have to look at loop as no function calls

within while loop, constant number of steps

how many times through loop?

- how many times can one divide i by 10?
- *O(log(i))*

LINEAR COMPLEXITY

- saw this last time
  - searching a list in sequence to see if an element is present
  - iterative loops

O() FOR ITERATIVE FACTORIAL

- complexity can depend on number of iterative calls

```
def fact_iter(n):
    prod = 1
    for i in range(1, n+1):
        prod *= i
    return prod
```

- overall ***O(n)*** – n times round loop, constant cost each time

O() FOR RECURSIVE FACTORIAL

```
def fact_recur(n):
    """ assume n >= 0 """
    if n <= 1:
        return 1
    else:
        return n*fact_recur(n – 1)
```

- computes factorial recursively
- if you time it, may notice that it runs a bit slower than iterative version due to function calls
- still ***O(n)*** because the number of function calls is linear in n, and constant effort to set up call
- **iterative and recursive factorial** implementations are the **same order of growth**

LOG-LINEAR COMPLEITY

- many practical algorithms are log-linear
- very commonly used log-linear algorithm is merge sort
- will return to this next lecture

POLYNOMIAL COMPLEXITY

- most common polynomial algorithms are quadratic, i.e., complexity grows with square of size of input
- commonly occurs when we have nested loops or recursive function calls
- saw this last time

EXPONENTIAL COMPLEXITY

- recursive functions where more than one recursive call for each size of problem
  - Towers of Hanoi
- many important problems are inherently exponential
  - unfortunate, as cost can be high
  - will lead us to consider approximate solutions as may provide reasonable answer more quickly

COMPLEXITY OF TOWERS OF HANOI

- Let t_n denote time to solve tower of size n
- $t_n = 2t_{n-1} + 1$
- $= 2(2t_{n-2} + 1) + 1$
- $= 4t_{n-2} + 2 + 1$
- $= 4(2t_{n-3} + 1) + 2 + 1$
- $= 8t_{n-3} + 4 + 2 + 1$
- $= 2^k t_{n-k} + 2^{k-1} + \ldots + 4 + 2 + 1$
- $= 2^{n-1} + 2^{n-2} + \ldots + 4 + 2 + 1$
- $= 2^n – 1$
- so order of growth is *$O(2^n)$*

Geometric growth

$$
\begin{aligned}
a &= \phantom{2^n +{}} 2^{n-1} + \ldots + 2 + 1 \\
2a &= 2^n + 2^{n-1} + \ldots + 2 \\
a &= 2^n \phantom{+ 2^{n-1} + \ldots + 2 + 1} - 1
\end{aligned}
$$

EXPONENTIAL COMPLEXITY

- given a set of integers (with no repeats), want to generate the collection of all possible subsets – called the power set
- {1, 2, 3, 4} would generate
    - {}, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}
- order doesn't matter
    - {}, {1}, {2}, {1, 2}, {3}, {1, 3}, {2, 3}, {1, 2, 3}, {4}, {1, 4}, {2, 4}, {1, 2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}

POWER SET – CONCEPT

- we want to generate the power set of integers from 1 to n
- assume we can generate power set of integers from 1 to n-1
- then all of those subsets belong to bigger power set (choosing not include n); and all of those subsets with n added to each of them also belong to the bigger power set (choosing to include n)
- {}, {1}, {2}, {1, 2}, {3}, {1, 3}, {2, 3}, {1, 2, 3}, {4}, {1, 4}, {2, 4}, {1, 2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}
- nice recursive description!

EXPONENTIAL COMPLEXITY

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]] #list of empty list
    smaller = genSubsets(L[:-1]) # all subsets without
 last element
    extra = L[-1:] # create a list of just last element
    new = []
    for small in smaller:
        new.append(small+extra)  # for all smaller
 solutions, add one with last element
    return smaller+new  # combine those with last
 element and those without
```

EXPONENTIAL COMPLEXITY

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]]
    smaller = genSubsets(L[:-1])
    extra = L[-1:]
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

assuming append is constant time

time includes time to solve smaller problem, plus time needed to make a copy of all elements in smaller problem

EXPONENTIAL COMPLEXITY

```
def genSubsets(L):
    res = []
    if len(L) == 0:
        return [[]]
    smaller = genSubsets(L[:-1])
    extra = L[-1:]
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

but important to think about size of smaller

know that for a set of size k there are 2^k cases

how can we deduce overall complexity?

EXPONENTIAL COMPLEXITY

- let t_n denote time to solve problem of size n
- let s_n denote size of solution for problem of size n
- $t_n = t_{n-1} + s_{n-1} + c$ (where c is some constant number of operations)
- $t_n = t_{n-1} + 2^{n-1} + c$
- $= t_{n-2} + 2^{n-2} + c + 2^{n-1} + c$
- $= t_{n-k} + 2^{n-k} + \ldots + 2^{n-1} + kc$
- $= t_0 + 2^0 + \ldots + 2^{n-1} + nc$
- $= 1 + 2^n + nc$

Thus computing power set is ***$O(2^n)$***

COMPLEXITY CLASSES

- *O(1)* – code does not depend on size of problem
- *O(log n)* – reduce problem in half each time through process
- *O(n)* – simple iterative or recursive programs
- *O(n log n)* – will see next time
- $O(n^c)$ – nested loops or recursive calls
- $O(c^n)$ – multiple recursive calls at each level

SOME MORE EXAMPLES OF ANALYZING COMPLEXITY

COMPLEXITY OF ITERATIVE FIBONACCI

```
def fib_iter(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        fib_i = 0
        fib_ii = 1
        for i in range(n-1):
            tmp = fib_i
            fib_i = fib_ii
            fib_ii = tmp + fib_ii
        return fib_ii
```

constant
O(1)

constant
O(1)

linear
O(n)

constant
O(1)

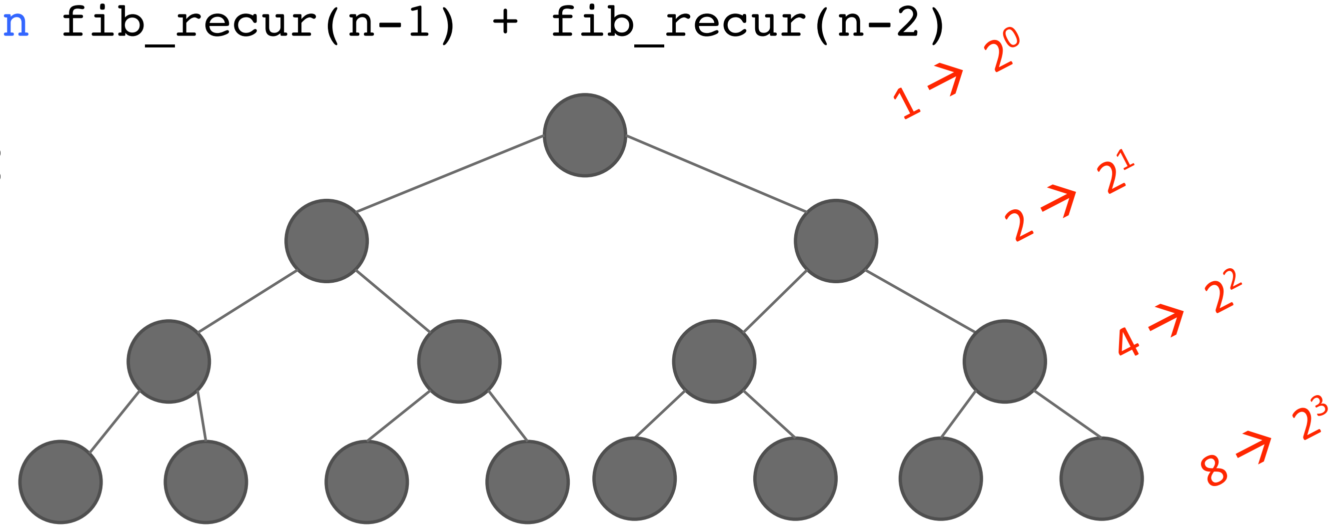
- Best case:
  O(1)
- Worst case:
  O(1) + O(n) + O(1) → **O(n)**

COMPLEXITY OF RECURSIVE FIBONACCI

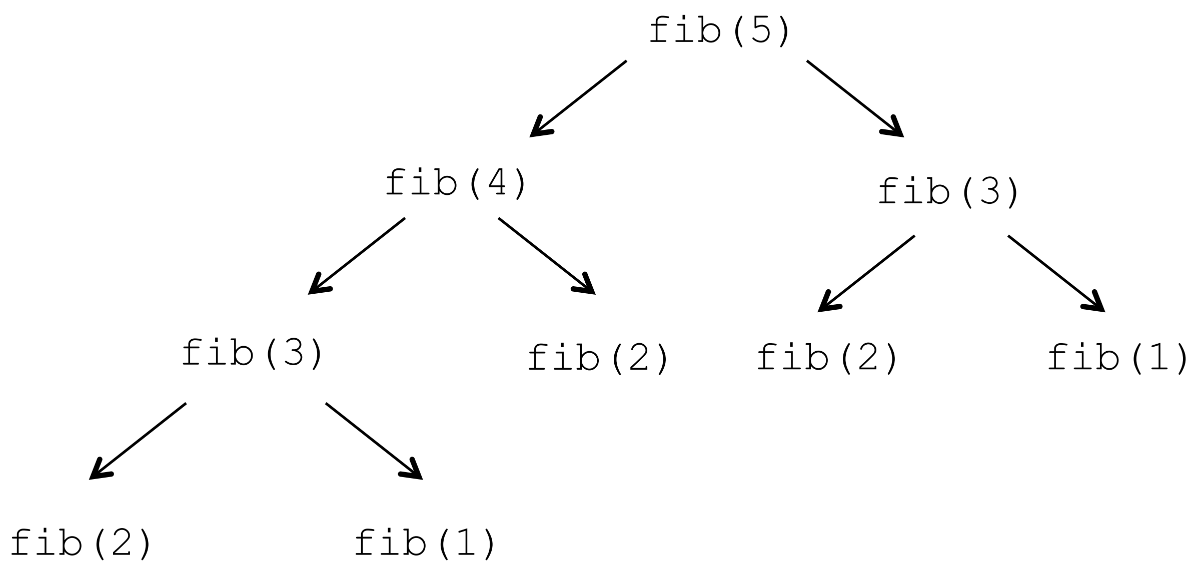
```
def fib_recur(n):
    """ assumes n an int >= 0 """
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib_recur(n-1) + fib_recur(n-2)
```

- Worst case:

  $O(2^n)$



COMPLEXITY OF RECURSIVE FIBONACCI



- actually can do a bit better than 2^n since tree of cases thins out to right
- but complexity is still exponential

BIG OH SUMMARY

- compare **efficiency of algorithms**
  - notation that describes growth
  - **lower order of growth** is better
  - independent of machine or specific implementation

- use Big Oh
  - describe order of growth
  - **asymptotic notation**
  - **upper bound**
  - **worst case** analysis

COMPLEXITY OF COMMON PYTHON FUNCTIONS

- Lists: `n` is `len(L)`
  - index O(1)
  - store O(1)
  - length O(1)
  - append O(1)
  - == O(n)
  - remove O(n)
  - copy O(n)
  - reverse O(n)
  - iteration O(n)
  - in list O(n)

- Dictionaries: `n` is `len(d)`
- worst case
  - index O(n)
  - store O(n)
  - length O(n)
  - delete O(n)
  - iteration O(n)
- average case
  - index O(1)
  - store O(1)
  - delete O(1)
  - iteration O(n)