6.003: Signals and Systems

Convolution

October 4, 2011

Mid-term Examination #1

Wednesday, October 5, 7:30-9:30pm.

No recitations on the day of the exam.

Coverage: CT and DT Systems, Z and Laplace Transforms Lectures 1–7 Recitations 1–7 Homeworks 1–4

Homework 4 will not collected or graded. Solutions are posted.

Closed book: 1 page of notes $(8\frac{1}{2} \times 11 \text{ inches}; \text{ front and back}).$

No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.

Prior term midterm exams have been posted on the 6.003 website.

Multiple Representations of CT and DT Systems

Verbal descriptions: preserve the rationale.

Difference/differential equations: mathematically compact.

$$y[n] = x[n] + z_0 y[n-1]$$
 $\dot{y}(t) = x(t) + s_0 y(t)$

Block diagrams: illustrate signal flow paths.



Operator representations: analyze systems as polynomials.

$$\frac{Y}{X} = \frac{1}{1 - z_0 \mathcal{R}} \qquad \qquad \frac{Y}{X} = \frac{\mathcal{A}}{1 - s_0 \mathcal{A}}$$

Transforms: representing diff. equations with algebraic equations.

$$H(z) = \frac{z}{z - z_0} \qquad \qquad H(s) = \frac{1}{s - s_0}$$

Convolution

Representing a system by a single signal.

Although we have focused on responses to simple signals ($\delta[n], \delta(t)$) we are generally interested in responses to more complicated signals.

How do we compute responses to a more complicated input signals?

No problem for difference equations / block diagrams.

 \rightarrow use step-by-step analysis.

Check Yourself









































Check Yourself



Superposition

Break input into additive parts and sum the responses to the parts.



Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given
$$x_1[n] \longrightarrow \text{system} \longrightarrow y_1[n]$$
 and $x_2[n] \longrightarrow \text{system} \longrightarrow y_2[n]$

the system is linear if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \text{system} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β .

Superposition

Break input into additive parts and sum the responses to the parts.



Superposition works if the system is linear.

Superposition

Break input into additive parts and sum the responses to the parts.



Reponses to parts are easy to compute if system is **time-invariant**.

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

$$x[n] \longrightarrow$$
 system $\longrightarrow y[n]$

the system is time invariant if

$$x[n-n_0] \longrightarrow$$
 system $\longrightarrow y[n-n_0]$

is true for all n_0 .

Superposition

Break input into additive parts and sum the responses to the parts.



Superposition is easy if the system is linear and time-invariant.

Structure of Superposition

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit-sample responses.

$$\delta[n] \longrightarrow \text{system} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \text{system} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \text{system} \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \text{system} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution

Response of an LTI system to an arbitrary input.

$$x[n] \longrightarrow \text{LTI} \longrightarrow y[n]$$
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$$

This operation is called **convolution**.

Notation

Convolution is represented with an asterisk.

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$$

It is customary (but confusing) to abbreviate this notation: (x * h)[n] = x[n] * h[n]

Notation

Do not be fooled by the confusing notation.

Confusing (but conventional) notation:

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

x[n] * h[n] looks like an operation of samples; but it is not!

 $x[1]*h[1]\neq (x*h)[1]$

Convolution operates on signals not samples.

Unambiguous notation:

 $\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$

The symbols x and h represent DT signals. Convolving x with h generates a new DT signal x * h.



















$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

flip *



h[k]C О **o**- k 3 $-2-1 \ 0 \ 1$ 254 h[-k]C O O **o-** k 1 $-2 - 1 \, 0$ 253 4

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$





$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



$$y[0] = \sum_{k=-\infty}^{\infty} \boldsymbol{x}[k]\boldsymbol{h}[0-k]$$



○- k

o- k

5

5



















o- k

- k

5


Structure of Convolution



Structure of Convolution





Express mathematically:

$$\left(\left(\frac{2}{3}\right)^{n} u[n]\right) * \left(\left(\frac{2}{3}\right)^{n} u[n]\right) = \sum_{k=-\infty}^{\infty} \left(\left(\frac{2}{3}\right)^{k} u[k]\right) \times \left(\left(\frac{2}{3}\right)^{n-k} u[n-k]\right)$$
$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^{k} \times \left(\frac{2}{3}\right)^{n-k}$$
$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^{n} = \left(\frac{2}{3}\right)^{n} \sum_{k=0}^{n} 1$$
$$= (n+1) \left(\frac{2}{3}\right)^{n} u[n]$$
$$= 1, \ \frac{4}{3}, \ \frac{4}{3}, \ \frac{32}{27}, \ \frac{80}{81}, \ \dots$$



DT Convolution: Summary

Representing an LTI system by a single signal.

$$x[n] \longrightarrow h[n] \longrightarrow y[n]$$

Unit-sample response h[n] is a complete description of an LTI system.

Given h[n] one can compute the response y[n] to any arbitrary input signal x[n]:

$$y[n] = (x * h)[n] \equiv \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CT Convolution

The same sort of reasoning applies to CT signals.



As
$$\Delta \to 0$$
, $k\Delta \to \tau$, $\Delta \to d\tau$, and $p(t) \to \delta(t)$:
 $x(t) \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

Structure of Superposition

x(t)

If a system is linear and time-invariant (LTI) then its output is the integral of weighted and shifted unit-impulse responses.

$$\delta(t) \longrightarrow \text{system} \rightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow \text{system} \rightarrow h(t-\tau)$$

$$x(\tau)\delta(t-\tau) \longrightarrow \text{system} \rightarrow x(\tau)h(t-\tau)$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \text{system} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

CT Convolution

Convolution of CT signals is analogous to convolution of DT signals.

DT:
$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

CT:
$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$



Which plot shows the result of the following convolution?





Convolution

Convolution is an important **computational tool**.

Example: characterizing LTI systems

- Determine the unit-sample response h[n].
- Calculate the output for an arbitrary input using convolution:

$$y[n] = (x * h)[n] = \sum x[k]h[n-k]$$

Applications of Convolution

Convolution is an important **conceptual tool:** it provides an important new way to **think** about the behaviors of systems.

Example systems: microscopes and telescopes.

Images from even the best microscopes are blurred.



Image by MIT OpenCourseWare.

A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



Blurring can be represented by convolving the image with the optical "point-spread-function" (3D impulse response).



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Measuring the "impulse response" of a microscope.

Image diameter ≈ 6 times target diameter: target \rightarrow impulse.



image of a 0.09 $\mu \mathrm{m}$ bead

images by Anthony Patire

Courtesy of Anthony Patire. Used with permission.

Images at different focal planes can be assembled to form a threedimensional impulse response (point-spread function).



Courtesy of Anthony Patire. Used with permission.

Blurring along the optical axis is better visualized by resampling the three-dimensional impulse response.



Courtesy of Anthony Patire. Used with permission.

Blurring is much greater along the optical axis than it is across the optical axis.



Courtesy of Anthony Patire. Used with permission.

The point-spread function (3D impulse response) is a useful way to characterize a microscope. It provides a direct measure of blurring, which is an important figure of merit for optics.

Hubble Space Telescope (1990-)



Why build a space telescope?

Telescope images are blurred by the telescope lenses AND by atmospheric turbulence.



Telescope blur can be respresented by the convolution of blur due to atmospheric turbulence and blur due to mirror size.





The main optical components of the Hubble Space Telescope are two mirrors.



The diameter of the primary mirror is 2.4 meters.



Hubble's first pictures of distant stars (May 20, 1990) were more blurred than expected.



expected point-spread function early Hubble image of distant star

The parabolic mirror was ground 2.2 μ m too flat!



Corrective Optics Space Telescope Axial Replacement (COSTAR): eyeglasses for Hubble!



Hubble images before and after COSTAR.



before

after

Hubble images before and after COSTAR.



before

after

Images from ground-based telescope and Hubble.



Impulse Response: Summary

The impulse response is a complete description of a linear, timeinvariant system.

One can find the output of such a system by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems, e.g., optical systems, where blurring is an important figure of merit. MIT OpenCourseWare http://ocw.mit.edu

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