Lecture 14: Johnson’s Algorithm

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All-Pairs Shortest Paths (APSP)

- **Input:** directed graph $G = (V, E)$ with weights $w : E \to \mathbb{Z}$
- **Output:** $\delta(u, v)$ for all $u, v \in V$, or abort if $G$ contains negative-weight cycle
- Useful when understanding whole network, e.g., transportation, circuit layout, supply chains...
- Just doing a SSSP algorithm $|V|$ times is actually pretty good, since output has size $O(|V|^2)$
  - $|V| \cdot O(|V| + |E|)$ with BFS if weights positive and bounded by $O(|V| + |E|)$
  - $|V| \cdot O(|V| + |E|)$ with DAG Relaxation if acyclic
  - $|V| \cdot O(|V| \log |V| + |E|)$ with Dijkstra if weights non-negative or graph undirected
  - $|V| \cdot O(|V| \cdot |E|)$ with Bellman-Ford (general)
- **Today:** Solve APSP in any weighted graph in $|V| \cdot O(|V| \log |V| + |E|)$ time
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Approach

- **Idea:** Make all edge weights non-negative while preserving shortest paths!
- i.e., reweight $G$ to $G'$ with no negative weights, where a shortest path in $G$ is shortest in $G'$
- If non-negative, then just run Dijkstra $|V|$ times to solve APSP
- **Claim:** Can compute distances in $G$ from distances in $G'$ in $O(|V|(|V| + |E|))$ time
  - Compute shortest-path tree from distances, for each $s \in V'$ in $O(|V| + |E|)$ time (L11)
  - Also shortest-paths tree in $G$, so traverse tree with DFS while also computing distances
  - Takes $O(|V| \cdot (|V| + |E|))$ time (which is less time than $|V|$ times Dijkstra)
- But how to make $G'$ with non-negative edge weights? Is this even possible??
- **Claim:** Not possible if $G$ contains a negative-weight cycle
- **Proof:** Shortest paths are simple if no negative weights, but not if negative-weight cycle
- Given graph $G$ with negative weights but no negative-weight cycles, can we make edge weights non-negative while preserving shortest paths?

Making Weights Non-negative

- **Idea!** Add negative of smallest weight in $G$ to every edge! All weights non-negative! :)
- **FAIL:** Does not preserve shortest paths! Biases toward paths traversing fewer edges :
- **Idea!** Given vertex $v$, add $h$ to all outgoing edges and subtract $h$ from all incoming edges
- **Claim:** Shortest paths are preserved under the above reweighting
- **Proof:**
  - Weight of every path starting at $v$ changes by $h$
  - Weight of every path ending at $v$ changes by $-h$
  - Weight of a path passing through $v$ does not change (locally)
- This is a very general and useful trick to transform a graph while preserving shortest paths!
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- Even works with multiple vertices!
- Define a potential function \( h : V \rightarrow \mathbb{Z} \) mapping each vertex \( v \in V \) to a potential \( h(v) \)
- Make graph \( G' \): same as \( G \) but edge \((u, v) \in E\) has weight \( w'(u, v) = w(u, v) + h(u) - h(v) \)
- **Claim:** Shortest paths in \( G \) are also shortest paths in \( G' \)
  - **Proof:**
    - Weight of path \( \pi = (v_0, \ldots, v_k) \) in \( G \) is \( w(\pi) = \sum_{i=1}^{k} w(v_{i-1}, v_i) \)
    - Weight of \( \pi \) in \( G' \) is: \( \sum_{i=1}^{k} w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i) = w(\pi) + h(v_0) - h(v_k) \)
    - (Sum of \( h \)'s telescope, since there is a positive and negative \( h(v_i) \) for each interior \( i \))
    - Every path from \( v_0 \) to \( v_k \) changes by the same amount
    - So any shortest path will still be shortest

Making Weights Non-negative

- Can we find a potential function such that \( G' \) has no negative edge weights?
  - i.e., is there an \( h \) such that \( w(u, v) + h(u) - h(v) \geq 0 \) for every \((u, v) \in E\)?
  - Re-arrange this condition to \( h(v) \leq h(u) + w(u, v) \), looks like **triangle inequality**!
- **Idea!** Condition would be satisfied if \( h(v) = \delta(s, v) \) and \( \delta(s, v) \) is finite for some \( s \)
- But graph may be disconnected, so may not exist any such vertex \( s \)
  - **Idea!** Add a new vertex \( s \) with a directed 0-weight edge to every \( v \in V \)
- \( \delta(s, v) \leq 0 \) for all \( v \in V \), since path exists a path of weight 0
- **Claim:** If \( \delta(s, v) = -\infty \) for any \( v \in V \), then the original graph has a negative-weight cycle
  - **Proof:**
    - Adding \( s \) does not introduce new cycles (\( s \) has no incoming edges)
    - So if reweighted graph has a negative-weight cycle, so does the original graph
  - Alternatively, if \( \delta(s, v) \) is finite for all \( v \in V \):
    - \( w'(u, v) = w(u, v) + h(u) - h(v) \geq 0 \) for every \((u, v) \in E\) by triangle inequality!
    - New weights in \( G' \) are non-negative while preserving shortest paths!
Johnson’s Algorithm

- Construct $G_x$ from $G$ by adding vertex $x$ connected to each vertex $v \in V$ with 0-weight edge
- Compute $\delta_x(x, v)$ for every $v \in V$ (using Bellman-Ford)
- If $\delta_x(x, v) = -\infty$ for any $v \in V$:
  - Abort (since there is a negative-weight cycle in $G$)
- Else:
  - Reweight each edge $w'(u, v) = w(u, v) + \delta_x(x, u) - \delta_x(x, v)$ to form graph $G'$
  - For each $u \in V$:
    * Compute shortest-path distances $\delta'(u, v)$ to all $v$ in $G'$ (using Dijkstra)
    * Compute $\delta(u, v) = \delta'(u, v) - \delta_x(x, u) + \delta_x(x, v)$ for all $v \in V$

Correctness

- Already proved that transformation from $G$ to $G'$ preserves shortest paths
- Rest reduces to correctness of Bellman-Ford and Dijkstra
- Reducing from Signed APSP to Non-negative APSP
- Reductions save time! No induction today! :) 

Running Time

- $O(|V| + |E|)$ time to construct $G_x$
- $O(|V||E|)$ time for Bellman-Ford
- $O(|V| + |E|)$ time to construct $G'$
- $O(|V| \cdot (|V| \log |V| + |E|))$ time for $|V|$ runs of Dijkstra
- $O(|V|^2)$ time to compute distances in $G$ from distances in $G'$
- $O(|V|^2 \log |V| + |V||E|)$ time in total