Lecture 15: Recursive Algorithms

How to Solve an Algorithms Problem (Review)

- Reduce to a problem you already know (use data structure or algorithm)

Search Data Structures	Sort Algorithms	Graph Algorithms
Array	Insertion Sort	Breadth First Search
Linked List	Selection Sort	DAG Relaxation (DFS + Topo)
Dynamic Array	Merge Sort	Dijkstra
Sorted Array	Counting Sort	Bellman-Ford
Direct-Access Array	Radix Sort	Johnson
Hash Table	AVL Sort	
AVL Tree	Heap Sort	
Binary Heap		

- Design your own **recursive** algorithm
  - Constant-sized program to solve arbitrary input
  - Need looping or recursion, analyze by induction
  - Recursive function call: vertex in a graph, directed edge from $A \to B$ if B calls A
  - Dependency graph of recursive calls must be acyclic (if can terminate)
  - Classify based on shape of graph

Class	Graph
Brute Force	Star
Decrease & Conquer	Chain
Divide & Conquer	Tree
Dynamic Programming	DAG
Greedy/Incremental	Subgraph

  - Hard part is thinking inductively to construct recurrence on subproblems
  - How to solve a problem recursively (**SRT BOT**)
    1. **Subproblem** definition
    2. **Relate** subproblem solutions recursively
    3. **Topological order** on subproblems ($\Rightarrow$ subproblem DAG)
    4. **Base** cases of relation
    5. **Original** problem solution via subproblem(s)
    6. **Time** analysis

Merge Sort in SRT BOT Framework

- Merge sorting an array A of n elements can be expressed in SRT BOT as follows:
  - **S**ubproblems: $S(i, j)$ = sorted array on elements of $A[i : j]$ for $0 \leq i \leq j \leq n$
  - **R**elation: $S(i, j) = \text{merge}(S(i, m), S(m, j))$ where $m = \lfloor (i + j)/2 \rfloor$
  - **T**opo. order: Increasing $j - i$
  - **B**ase cases: $S(i, i + 1) = [A[i]]$
  - **O**riginal: $S(0, n)$
  - **T**ime: $T(n) = 2\,T(n/2) + O(n) = O(n \lg n)$
- In this case, subproblem DAG is a tree (divide & conquer)

Fibonacci Numbers

- Suppose we want to compute the nth Fibonacci number F_n
- **S**ubproblems: $F(i)$ = the ith Fibonacci number F_i for $i \in \{0, 1, \ldots, n\}$
- **R**elation: $F(i) = F(i - 1) + F(i - 2)$ (definition of Fibonacci numbers)
- **T**opo. order: Increasing i
- **B**ase cases: $F(0) = 0$, $F(1) = 1$
- **O**riginal prob.: $F(n)$

```python
def fib(n):
    if n < 2: return n                  # base case
    return fib(n - 1) + fib(n - 2)      # recurrence
```

- Divide and conquer implies a tree of **recursive calls** (draw tree)
- **T**ime: $T(n) = T(n-1) + T(n-2) + O(1) > 2T(n-2)$, $T(n) = \Omega(2^{n/2})$ exponential... :(
- Subproblem $F(k)$ computed more than once! ($F(n - k)$ times)
- Can we avoid this waste?

Re-using Subproblem Solutions

- Draw subproblem dependencies as a DAG
- To solve, either:
  - **Top down:** record subproblem solutions in a memo and re-use (**recursion + memoization**)
  - **Bottom up:** solve subproblems in topological sort order (usually via loops)
- For Fibonacci, $n + 1$ subproblems (vertices) and $< 2n$ dependencies (edges)
- Time to compute is then $O(n)$ additions

```
# recursive solution (top down)
def fib(n):
    memo = {}
    def F(i):
        if i < 2: return i                  # base cases
        if i not in memo:                   # check memo
            memo[i] = F(i - 1) + F(i - 2)   # relation
        return memo[i]
    return F(n)                             # original
```

```
# iterative solution (bottom up)
def fib(n):
    F = {}
    F[0], F[1] = 0, 1                       # base cases
    for i in range(2, n + 1):               # topological order
        F[i] = F[i - 1] + F[i - 2]          # relation
    return F[n]                             # original
```

- A subtlety is that Fibonacci numbers grow to $\Theta(n)$ bits long, potentially $\gg$ word size w
- Each addition costs $O(\lceil n/w \rceil)$ time
- So total cost is $O(n\lceil n/w \rceil) = O(n + n^2/w)$ time

Dynamic Programming

- Weird name coined by Richard Bellman
  - Wanted government funding, needed cool name to disguise doing mathematics!
  - Updating (dynamic) a plan or schedule (program)
- Existence of recursive solution implies decomposable subproblems[1]
- Recursive algorithm implies a graph of computation
- Dynamic programming if subproblem dependencies **overlap** (DAG, in-degree > 1)
- "Recurse but re-use" (Top down: record and lookup subproblem solutions)
- "Careful brute force" (Bottom up: do each subproblem in order)
- Often useful for **counting/optimization** problems: almost trivially correct recurrences

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How to Solve a Problem Recursively (SRT BOT)

1. **Subproblem** definition   subproblem $x \in X$
   - Describe the meaning of a subproblem **in words**, in terms of parameters
   - Often subsets of input: prefixes, suffixes, contiguous substrings of a sequence
   - Often record partial state: add subproblems by incrementing some auxiliary variables
2. **Relate** subproblem solutions recursively $x(i) = f(x(j), \ldots)$ for one or more $j < i$
3. **Topological order** to argue relation is acyclic and subproblems form a DAG
4. **Base** cases
   - State solutions for all (reachable) independent subproblems where relation breaks down
5. **Original problem**
   - Show how to compute solution to original problem from solutions to subproblem(s)
   - Possibly use parent pointers to recover actual solution, not just objective function
6. **Time** analysis
   - $\sum_{x \in X} \text{work}(x)$, or if $\text{work}(x) = O(W)$ for all $x \in X$, then $|X| \cdot O(W)$
   - $\text{work}(x)$ measures **nonrecursive** work in relation; treat recursions as taking $O(1)$ time

[1] This property often called **optimal substructure**. It is a property of recursion, not just dynamic programming

DAG Shortest Paths

- Recall the DAG SSSP problem: given a DAG G and vertex s, compute $\delta(s, v)$ for all $v \in V$
- **S**ubproblems: $\delta(s, v)$ for all $v \in V$
- **R**elation: $\delta(s, v) = \min\{\delta(s, u) + w(u, v) \mid u \in \text{Adj}^-(v)\} \cup \{\infty\}$
- **T**opo. order: Topological order of G
- **B**ase cases: $\delta(s, s) = 0$
- **O**riginal: All subproblems
- **T**ime: $\sum_{v \in V} O(1 + |\text{Adj}^-(v)|) = O(|V| + |E|)$
- DAG Relaxation computes the same min values as this dynamic program, just
  - step-by-step (if new value $<$ min, update min via edge relaxation), and
  - from the perspective of u and $\text{Adj}^+(u)$ instead of v and $\text{Adj}^-(v)$

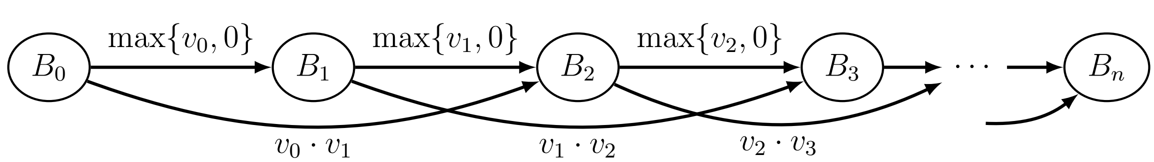
Bowling

- Given n pins labeled $0, 1, \ldots, n-1$
- Pin i has **value** v_i
- Ball of size similar to pin can hit either
  - 1 pin i, in which case we get v_i points
  - 2 adjacent pins i and $i+1$, in which case we get $v_i \cdot v_{i+1}$ points
- Once a pin is hit, it can't be hit again (removed)
- Problem: Throw zero or more balls to maximize total points
- Example: $[\,-1, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{9, 9}, \boxed{3}, \boxed{-3, -5}, \boxed{2, 2}\,]$

Bowling Algorithms

- Let's start with a more familiar divide-and-conquer algorithm:
  - **S**ubproblems: $B(i, j)$ = maximum score starting with just pins $i, i+1, \ldots, j-1$, for $0 \leq i \leq j \leq n$
  - **R**elation:
    - $m = \lfloor (i+j)/2 \rfloor$
    - Either hit m and $m+1$ together, or don't
    - $B(i, j) = \max\{v_m \cdot v_{m+1} + B(i, m) + B(m+2, j), B(i, m+1) + B(m+1, j)\}$
  - **T**opo. order: Increasing $j - i$
  - **B**ase cases: $B(i, i) = 0$, $B(i, i+1) = \max\{v_i, 0\}$
  - **O**riginal: $B(0, n)$
  - **T**ime: $T(n) = 4\,T(n/2) + O(1) = O(n^2)$
- This algorithm works but isn't very fast, and doesn't generalize well (e.g., to allow for a bigger ball that hits three balls at once)

- Dynamic programming algorithm: use suffixes
  - **S**ubproblems: $B(i)$ = maximum score starting with just pins $i, i+1, \ldots, n-1$, for $0 \leq i \leq n$
  - **R**elation:
    - Locally brute-force what could happen with first pin (original pin i): skip pin, hit one pin, hit two pins
    - Reduce to smaller suffix and recurse, either $B(i+1)$ or $B(i+2)$
    - $B(i) = \max\{B(i+1), v_i + B(i+1), v_i \cdot v_{i+1} + B(i+2)\}$
  - **T**opo. order: Decreasing i (for $i = n, n-1, \ldots, 0$)
  - **B**ase cases: $B(n) = B(n+1) = 0$
  - **O**riginal: $B(0)$
  - **T**ime: (assuming memoization)
    - $\Theta(n)$ subproblems $\cdot$ $\Theta(1)$ work in each
    - $\Theta(n)$ total time
- Fast and easy to generalize!
- Equivalent to maximum-weight path in Subproblem DAG:



Bowling Code

- Converting a SRT BOT specification into code is automatic/straightforward
- Here's the result for the Bowling Dynamic Program above:

```
# recursive solution (top down)
def bowl(v):
    memo = {}
    def B(i):
        if i >= len(v): return 0          # base cases
        if i not in memo:                 # check memo
            memo[i] = max(B(i+1),         # relation: skip pin i
                v[i] + B(i+1),            # OR bowl pin i separately
                v[i] * v[i+1] + B(i+2))   # OR bowl pins i and i+1 together
        return memo[i]
    return B(0)                           # original
```

```
# iterative solution (bottom up)
def bowl(v):
    B = {}
    B[len(v)] = 0                         # base cases
    B[len(v)+1] = 0
    for i in reversed(range(len(v))):     # topological order
        B[i] = max(B[i+1],                # relation: skip pin i
            v[i] + B(i+1),                # OR bowl pin i separately
            v[i] * v[i+1] + B(i+2))       # OR bowl pins i and i+1 together
    return B[0]                           # original
```

How to Relate Subproblem Solutions

- The general approach we're following to define a relation on subproblem solutions:
  - Identify a question about a subproblem solution that, if you knew the answer to, would reduce to "smaller" subproblem(s)
    - In case of bowling, the question is "how do we bowl the first couple of pins?"
  - Then locally brute-force the question by trying all possible answers, and taking the best
    - In case of bowling, we take the max because the problem asks to maximize
  - Alternatively, we can think of correctly guessing the answer to the question, and directly recursing; but then we actually check all possible guesses, and return the "best"
- The key for efficiency is for the question to have a small (polynomial) number of possible answers, so brute forcing is not too expensive
- Often (but not always) the nonrecursive work to compute the relation is equal to the number of answers we're trying

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