CHAPTER 5

Properties of LTI State-Space Models

5.1 INTRODUCTION

In Chapter 4 we introduced state-space models for dynamical systems. In this chapter we study the structure and solutions of LTI state-space models. Throughout the discussion we restrict ourselves to the single-input, single-output Lth-order CT LTI state-space model

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t) \tag{5.1}$$

$$y(t) = \mathbf{c}^T\mathbf{q}(t) + \mathtt{d}x(t) \ , \tag{5.2}$$

or the DT LTI state-space model

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \tag{5.3}$$

$$y[n] = \mathbf{c}^T\mathbf{q}[n] + \mathtt{d}x[n] \ . \tag{5.4}$$

Equation (5.1) constitutes a representation of CT LTI system dynamics in the form of a set of coupled, first-order, linear, constant-coefficient differential equations for the L variables in $\mathbf{q}(t)$, driven by the input $x(t)$. Equation (5.3) gives a similar difference-equation representation of DT LTI system dynamics.

The basic approach to analyzing LTI state-space models parallels what you should already be familiar with from solving linear constant-coefficient differential or difference equations (of any order) in one variable. Specifically, we first consider the zero-input response to nonzero initial conditions at some starting time, and then augment that with the response due to the nonzero input when the initial conditions are zero. Understanding the full solution from the starting time onwards will give us insight into system stability, and into how the internal behavior relates to the input-output characteristics of the system.

5.2 THE ZERO-INPUT RESPONSE AND MODAL REPRESENTATION

We take our starting time to be 0, without loss of generality (since we are dealing with time-invariant models). Consider the response of the undriven system corresponding to (5.1), i.e., the response with $x(t) \equiv 0$ for $t \geq 0$, but with some nonzero initial condition $\mathbf{q}(0)$. This is the zero-input-response (ZIR) of the system (5.1), and is a solution of the undriven (or unforced or homogeneous) system

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) \ . \tag{5.5}$$

It is natural when analyzing an undriven LTI system to look for a solution in exponential form (essentially because exponentials have the unique property that shifting them is equivalent to scaling them, and undriven LTI systems are characterized by invariance to shifting and scaling of solutions). We accordingly look for a nonzero solution of the form

$$\mathbf{q}(t) = \mathbf{v}e^{\lambda t} \ , \quad \mathbf{v} \neq \mathbf{0} \ , \tag{5.6}$$

where each state variable is a scalar multiple of the same exponential $e^{\lambda t}$, with these scalar multiples assembled into the vector $\mathbf{v}$. (The boldface $\mathbf{0}$ at the end of the preceding equation denotes an L-component column vector whose entries are all 0 — we shall use $\mathbf{0}$ for any vectors or matrices whose entries are all 0, with the correct dimensions being apparent from the context. Writing $\mathbf{v} \neq \mathbf{0}$ signifies that at least one component of $\mathbf{v}$ is nonzero.)

Substituting (5.6) into (5.5) results in the equation

$$\lambda \mathbf{v} e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t} \ , \tag{5.7}$$

from which we can conclude that the vector $\mathbf{v}$ and scalar λ must satisfy

$$\lambda \mathbf{v} = \mathbf{A}\mathbf{v} \quad \text{or equivalently} \quad (\lambda \mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0} \ , \quad \mathbf{v} \neq \mathbf{0} \ , \tag{5.8}$$

where $\mathbf{I}$ denotes the identity matrix, in this case of dimension $L \times L$. The above equation has a nonzero solution $\mathbf{v}$ if and only if the coefficient matrix $(\lambda \mathbf{I} - \mathbf{A})$ is not invertible, i.e., if and only if its determinant is 0:

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0 \ . \tag{5.9}$$

For an Lth-order system, it turns out that the above determinant is a monic polynomial of degree L, called the characteristic polynomial of the system or of the matrix $\mathbf{A}$:

$$\det(\lambda I - \mathbf{A}) = a(\lambda) = \lambda^L + a_{L-1}\lambda^{L-1} + \cdots + a_0 \tag{5.10}$$

(The word "monic" simply means that the coefficient of the highest-degree term is 1.) It follows that (5.6) is a nonzero solution of (5.5) if and only if λ is one of the L roots $\{\lambda_i\}_{i=1}^{L}$ of the characteristic polynomial. These roots are referred to as characteristic roots of the system, and as eigenvalues of the matrix $\mathbf{A}$.

The vector $\mathbf{v}$ in (5.6) is correspondingly a nonzero solution $\mathbf{v}_i$ of the system of equations

$$(\lambda_i \mathbf{I} - \mathbf{A})\mathbf{v}_i = \mathbf{0} \ , \quad \mathbf{v}_i \neq \mathbf{0} \ , \tag{5.11}$$

and is termed the characteristic vector or eigenvector associated with λ_i. Note from (5.11) that multiplying any eigenvector by a nonzero scalar again yields an eigenvector, so eigenvectors are only defined up to a nonzero scaling. Any convenient scaling or normalization can be used.

In summary, the undriven system has a solution of the assumed exponential form in (5.6) if and only if λ equals some characteristic value or eigenvalue of $\mathbf{A}$, and the nonzero vector $\mathbf{v}$ is an associated characteristic vector or eigenvector.

We shall only be dealing with state-space models for which all the signals and the coefficient matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}^T$ and $\mathtt{d}$ are real-valued (though we may subsequently transform these models into the diagonal forms seen in the previous chapter, which may then have complex entries, but occurring in very structured ways). The coefficients a_i defining the characteristic polynomial $a(\lambda)$ in (5.10) are therefore real, and thus the complex roots of this polynomial occur in conjugate pairs. Also, it is straightforward to show that if $\mathbf{v}_i$ is an eigenvector associated with a complex eigenvalue λ_i, then $\mathbf{v}_i^*$ —i.e., the vector whose entries are the complex conjugates of the corresponding entries of $\mathbf{v}_i$ — is an eigenvector associated with λ_i^*, the complex conjugate of λ_i.

We refer to a nonzero solution of the form (5.6) for $\lambda = \lambda_i$ and $\mathbf{v} = \mathbf{v}_i$ as the ith mode of the system (5.1) or (5.5); the associated λ_i is termed the ith modal frequency or characteristic frequency or natural frequency of the system, and $\mathbf{v}_i$ is termed the ith mode shape. Note that if

$$\mathbf{q}(t) = \mathbf{v}_i e^{\lambda_i t} \tag{5.12}$$

then the corresponding initial condition must have been $\mathbf{q}(0) = \mathbf{v}_i$. It can be shown (though we don't do so here) that the system (5.5) — and similarly the system (5.1) — can only have one solution for a given initial condition, so it follows that for the initial condition $\mathbf{q}(0) = \mathbf{v}_i$, only the ith mode will be excited.

It can also be shown that eigenvectors associated with distinct eigenvalues are linearly independent, i.e., none of them can be written as a weighted linear combination of the remaining ones. For simplicity, we shall restrict ourselves throughout to the case where all L eigenvalues of $\mathbf{A}$ are distinct, which will guarantee that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_L$ form an independent set. (In some cases in which $\mathbf{A}$ has repeated eigenvalues, it is possible to find a full set of L independent eigenvectors, but this is not generally true.) We shall repeatedly use the fact that any vector in an L-dimensional space, such as our state vector $\mathbf{q}(t)$ at any specified time $t = t_0$, can be written as a unique linear combination of any L independent vectors in that space, such as our L eigenvectors.

5.2.1 Modal representation of the ZIR

Because (5.5) is linear, a weighted linear combination of modal solutions of the form (5.12), one for each eigenvalue, will also satisfy (5.5). Consequently a more general solution for the zero-input response with distinct eigenvalues is

$$\mathbf{q}(t) = \sum_{i=1}^{L} \alpha_i \mathbf{v}_i e^{\lambda_i t} \tag{5.13}$$

The expression in (5.13) can easily be verified to be a solution of (5.5) for arbitrary weights α_i, with initial condition

$$\mathbf{q}(0) = \sum_{i=1}^{L} \alpha_i \mathbf{v}_i \ . \tag{5.14}$$

Since the L eigenvectors $\mathbf{v}_i$ are independent under our assumption of L distinct eigenvalues, the right side of (5.14) can be made equal to any desired $\mathbf{q}(0)$ by proper choice of the coefficients α_i, and these coefficients are unique. Hence specifying the initial condition of the undriven system (5.5) specifies the α_i via (5.14), and thus specifies the full response of (5.5) via (5.13). In other words, (5.13) is actually a general expression for the ZIR of (5.1) — under our assumption of distinct eigenvalues. We refer to the expression on the right side of (5.13) as the modal decomposition of the ZIR.

The contribution to the modal decomposition from a conjugate pair of eigenvalues $\lambda_i = \sigma_i + j\omega_i$ and $\lambda_i^* = \sigma_i - j\omega_i$, with associated complex conjugate eigenvectors $\mathbf{v}_i = \mathbf{u}_i + j\mathbf{w}_i$ and $\mathbf{v}_i = \mathbf{u}_i - j\mathbf{w}_i$ respectively, will be a real term of the form

$$\alpha_i \mathbf{v}_i e^{\lambda_i t} + \alpha_i^* \mathbf{v}_i^* e^{\lambda_i^* t} \ . \tag{5.15}$$

With a little algebra, the real expression in (5.15) can be reduced to the form

$$\alpha_i \mathbf{v}_i e^{\lambda_i t} + \alpha_i^* \mathbf{v}_i^* e^{\lambda_i^* t} = K_i e^{\sigma_i t}[\mathbf{u}_i \cos(\omega_i t + \theta_i) - \mathbf{w}_i \sin(\omega_i t + \theta_i)] \tag{5.16}$$

for some constants K_i and θ_i that are determined by the initial conditions in the process of matching the two sides of (5.14). The above component of the modal solution therefore lies in the plane spanned by the real and imaginary parts, $\mathbf{u}_i$ and $\mathbf{w}_i$ respectively, of the eigenvector $\mathbf{v}_i$. The associated motion of the component of state trajectory in this plane involves an exponential spiral, with growth or decay of the spiral determined by whether $\sigma_i = \text{Re}\{\lambda_i\}$ is positive or negative respectively (corresponding to the eigenvalue λ_i — and its conjugate λ_i^* — lying in the open right- or left-half-plane respectively). If $\sigma_i = 0$, i.e., if the conjugate pair of eigenvalues lies on the imaginary axis, then the spiral degenerates to a closed loop. The rate of rotation of the spiral is determined by $\omega_i = \text{Im}\{\lambda_i\}$.

A similar development can be carried out in the DT case for the ZIR of (5.3). In that case (5.6) is replaced by a solution of the form

$$\mathbf{q}[n] = \mathbf{v}\lambda^n \tag{5.17}$$

and we find that when $\mathbf{A}$ has L distinct eigenvalues, the modal decomposition of the general ZIR solution takes the form

$$\mathbf{q}[n] = \sum_{i=1}^{L} \alpha_i \mathbf{v}_i \lambda_i^n \ . \tag{5.18}$$

5.2.2 Asymptotic stability

The stability of an LTI system is directly related to the behavior of the modes, and more specifically to the values of the λ_i, the roots of the characteristic polynomial. An LTI state-space system is termed asymptotically stable or internally stable if its ZIR decays to zero for all initial conditions. We see from (5.13) that the condition $\text{Re}\{\lambda_i\} < 0$ for all $1 \leq i \leq L$ is necessary and sufficient for asymptotic stability in the CT case. Thus, all eigenvalues of $\mathbf{A}$ in (5.1) — or natural frequencies of (5.1) — must be in the open left-half-plane.

In the DT case, (5.18) shows that a necessary and sufficient condition for asymptotic stability is $|\lambda_i| < 1$ for all $1 \leq i \leq L$, i.e., all eigenvalues of $\mathbf{A}$ in (5.3) — or natural frequencies of (5.3) — must be strictly within the unit circle.

We used the modal decompositions (5.13) and (5.18) to make these claims regarding stability conditions, but these modal decompositions were obtained under the assumption of distinct eigenvalues. Nevertheless, it can be shown that the stability conditions in the general case are identical to those above.

5.3 COORDINATE TRANSFORMATIONS

We have so far only described the zero-input response of LTI state-space systems. Before presenting the general response, including the effects of inputs, it will be helpful to understand how a given state-space representation can be transformed to an equivalent representation that might be simpler to analyze. Our development is carried out for the CT case, but an entirely similar development can be done for DT.

It is often useful to examine the behavior of a state-space system by rewriting the original description in terms of a transformed set of variables. A particularly important case involves the transformation of the state vector $\mathbf{q}(t)$ to a new state vector $\mathbf{r}(t)$ that decomposes the behavior of the system into its components along each of the eigenvectors $\mathbf{v}_i$:

$$\mathbf{q}(t) = \sum_{i=1}^{L} \mathbf{v}_i r_i(t) = \mathbf{V}\mathbf{r}(t) , \tag{5.19}$$

where the ith column of the $L \times L$ matrix $\mathbf{V}$ is the ith eigenvector, $\mathbf{v}_i$:

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_L \end{pmatrix} . \tag{5.20}$$

We refer to $\mathbf{V}$ as the modal matrix. Under our assumption of distinct eigenvalues, the eigenvectors are independent, which guarantees that $\mathbf{V}$ is invertible, so

$$\mathbf{r}(t) = \mathbf{V}^{-1}\mathbf{q}(t) . \tag{5.21}$$

The transformation from the original system description involving $\mathbf{q}(t)$ to one written in terms of $\mathbf{r}(t)$ is called a modal transformation, and the new state variables $r_i(t)$ defined through (5.19) are termed modal variables or modal coordinates.

More generally, a coordinate transformation corresponds to choosing a new state vector $\mathbf{z}(t)$ related to the original state vector $\mathbf{q}(t)$ through the relationship

$$\mathbf{q}(t) = \mathbf{M}\mathbf{z}(t) \tag{5.22}$$

where the constant matrix $\mathbf{M}$ is chosen to be invertible. (The ith column of $\mathbf{M}$ is the representation of the ith unit vector of the new $\mathbf{z}$ coordinates in terms of the old $\mathbf{q}$ coordinates.) Substituting (5.22) in (5.1) and (5.2), and solving for $\dot{\mathbf{z}}(t)$, we obtain

$$\dot{\mathbf{z}}(t) = (\mathbf{M}^{-1}\mathbf{A}\mathbf{M})\mathbf{z}(t) + (\mathbf{M}^{-1}\mathbf{b})x(t) \tag{5.23}$$

$$y(t) = (\mathbf{c}^T\mathbf{M})\mathbf{z}(t) + \mathtt{d}x(t)\ . \tag{5.24}$$

Equations (5.23) and (5.24) are still in state-space form, but with state vector $\mathbf{z}(t)$, and with modified coefficient matrices. This model is entirely equivalent to the original one, since (5.22) permits $\mathbf{q}(t)$ to be obtained from $\mathbf{z}(t)$, and the invertibility of $\mathbf{M}$ permits $\mathbf{z}(t)$ to be obtained from $\mathbf{q}(t)$. It is straightforward to verify that the eigenvalues of $\mathbf{A}$ are identical to those of $\mathbf{M}^{-1}\mathbf{A}\mathbf{M}$, and consequently that the natural frequencies of the transformed system are the same as those of the original system; only the eigenvectors change, with $\mathbf{v}_i$ transforming to $\mathbf{M}^{-1}\mathbf{v}_i$.

We refer to the transformation (5.22) as a similarity transformation, and say that the model (5.23), (5.24) is similar to the model (5.1), (5.2).

Note that the input $x(t)$ and output $y(t)$ are unaffected by this state transformation. For a given input, and assuming an initial state $\mathbf{z}(0)$ in the transformed system that is related to $\mathbf{q}(0)$ via (5.22), we obtain the same output as we would have from (5.1), (5.2). In particular, the transfer function from input to output is unaffected by a similarity transformation.

Similarity transformations can be defined in exactly the same way for the DT case in (5.3), (5.4).

5.3.1 Transformation to Modal Coordinates

What makes the modal similarity transformation (5.19) interesting and useful is the fact that the state evolution matrix $\mathbf{A}$ transforms to a diagonal matrix $\mathbf{\Lambda}$:

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \text{diagonal}\ \{\lambda_1, \cdots, \lambda_L\} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_L \end{bmatrix} = \mathbf{\Lambda}\ . \tag{5.25}$$

The easiest way to verify this is to establish the equivalent fact that $\mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}$, which in turn is simply the equation (5.11), written for $i = 1, \cdots, L$ and stacked up in matrix form.

The diagonal form of $\mathbf{\Lambda}$ causes the corresponding state equations in the new coordinate system to be decoupled. Under this modal transformation, the undriven system (5.5) is transformed into L decoupled, scalar equations:

$$\dot{r}_i(t) = \lambda_i r_i(t) \quad \text{for i} = 1, 2, \ldots, \text{L} \; . \tag{5.26}$$

Each of these is easy to solve:

$$r_i(t) = e^{\lambda_i t} r_i(0) \; . \tag{5.27}$$

Combining this with (5.19) yields (5.13) again, with $\alpha_i = r_i(0)$.

5.4 THE COMPLETE RESPONSE

Applying the modal transformation (5.19) to the full driven system (5.1), (5.2), we see that the transformed system (5.23), (5.24) takes the following form, which is decoupled into L parallel scalar subsystems:

$$\dot{r}_i(t) = \lambda_i r_i(t) + \beta_i x(t) \; , \quad i = 1, 2, \ldots, L \tag{5.28}$$

$$y(t) = \xi_1 r_1(t) + \cdots + \xi_L r_L(t) + \mathtt{d} x(t) \; , \tag{5.29}$$

where the β_i and ξ_i are defined via

$$\mathbf{V}^{-1}\mathbf{b} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_L \end{bmatrix} = \beta \; , \quad \mathbf{c}^T\mathbf{V} = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_L \end{bmatrix} = \xi \; . \tag{5.30}$$

The second equation in (5.30) shows that

$$\xi_i = \mathbf{c}^T \mathbf{v}_i \; . \tag{5.31}$$

To find an interpretation of the β_i, note that the first equation in (5.30) can be rewritten as $\mathbf{b} = \mathbf{V}\beta$. Writing out the product $\mathbf{V}\beta$ in detail, we find

$$\mathbf{b} = \mathbf{v}_1\beta_1 + \mathbf{v}_2\beta_2 + \cdots + \mathbf{v}_L\beta_L \; . \tag{5.32}$$

In other words, the coefficients β_i are the coefficients needed to express the input vector $\mathbf{b}$ as a linear combination of the eigenvectors $\mathbf{v}_i$.

Each of the scalar equations in (5.28) is a first-order LTI differential equation, and can be solved explicitly for $t \geq 0$, obtaining

$$r_i(t) = \underbrace{e^{\lambda_i t} r_i(0)}_{\text{ZIR}} + \underbrace{\int_0^t e^{\lambda_i(t-\tau)}\beta_i x(\tau)\, d\tau}_{\text{ZSR}} \; , \quad t \geq 0 \; , \quad 1 \leq i \leq L \; . \tag{5.33}$$

Expressed in this form, we easily recognize the separate contributions to the solution made by: (i) the response due to the initial state (the zero-input response or ZIR); and (ii) the response due to the system input (the zero-state response or ZSR). From the preceding expression and (5.29), one can obtain an expression for $y(t)$.

Introducing the natural "matrix exponential" notation

$$e^{\mathbf{\Lambda}t} = \text{diagonal}\ \{e^{\lambda_1 t}, \cdots, e^{\lambda_L t}\} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_L t} \end{bmatrix} \tag{5.34}$$

allows us to combine the L equations in (5.33) into the following single matrix equation:

$$\mathbf{r}(t) = e^{\mathbf{\Lambda}t}\mathbf{r}(0) + \int_0^t e^{\mathbf{\Lambda}(t-\tau)}\beta x(\tau)\, d\tau\ , \quad t \geq 0 \tag{5.35}$$

(where the integral of a vector is interpreted as the component-wise integral). Combining this equation with the expression (5.19) that relates $\mathbf{r}(t)$ to $\mathbf{q}(t)$, we finally obtain

$$\mathbf{q}(t) = \Big(\mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}\Big)\mathbf{q}(0) + \int_0^t \Big(\mathbf{V}e^{\mathbf{\Lambda}(t-\tau)}\mathbf{V}^{-1}\Big)\mathbf{b}x(\tau)\, d\tau \tag{5.36}$$

$$= e^{\mathbf{A}t}\mathbf{q}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}x(\tau)\, d\tau\ , \quad t \geq 0\ , \tag{5.37}$$

where, by analogy with (5.25), we have defined the matrix exponential

$$e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{\Lambda}t}\mathbf{V}^{-1}\ . \tag{5.38}$$

Equation (5.37) gives us, in compact matrix notation, the general solution of the CT LTI system (5.1).

An entirely parallel development can be carried out for the DT LTI case. The corresponding expression for the solution of (5.3) is

$$\mathbf{q}[n] = \Big(\mathbf{V}\mathbf{\Lambda}^n\mathbf{V}^{-1}\Big)\mathbf{q}[0] + \sum_{k=0}^{n-1}\Big(\mathbf{V}\mathbf{\Lambda}^{n-k-1}\mathbf{V}^{-1}\Big)\mathbf{b}x[k] \tag{5.39}$$

$$= \mathbf{A}^n\mathbf{q}[0] + \sum_{k=0}^{n-1}\mathbf{A}^{n-k-1}\mathbf{b}x[k]\ , \quad n \geq 0\ . \tag{5.40}$$

Equation (5.40) is exactly the expression one would get by simply iterating (5.3) forward one step at a time, to get $\mathbf{q}[n]$ from $\mathbf{q}[0]$. However, we get additional insight from writing the expression in the modally decomposed form (5.39), because it brings out the role of the eigenvalues of $\mathbf{A}$, i.e., the natural frequencies of the DT system, in determining the behavior of the system, and in particular its stability properties.

5.5 TRANSFER FUNCTION, HIDDEN MODES, REACHABILITY, OBSERVABILITY

The transfer function $H(s)$ of the transformed model (5.28), (5.29) describes the zero-state input-output relationship in the Laplace transform domain, and is straightforward to find because the equations are totally decoupled. Taking the Laplace transforms of those equations, with zero initial conditions in (5.28), results in

$$R_i(s) = \frac{\beta_i}{s - \lambda_i} X(s) \tag{5.41}$$

$$Y(s) = \Big(\sum_1^L \xi_i R_i(s) \Big) + \mathtt{d} X(s) \ . \tag{5.42}$$

Since $Y(s) = H(s)X(s)$, we obtain

$$H(s) = \Big(\sum_1^L \frac{\xi_i \beta_i}{s - \lambda_i} \Big) + \mathtt{d} \tag{5.43}$$

which can be rewritten in matrix notation as

$$H(s) = \xi^T (s\mathbf{I} - \mathbf{\Lambda})^{-1} \beta + \mathtt{d} \ . \tag{5.44}$$

This is also the transfer function of the original model in (5.1), (5.2), as similarity transformations do not change transfer functions. An alternative expression for the transfer function of (5.1), (5.2) follows from examination of the Laplace transformed version of (5.1), (5.2). We omit the details, but the resulting expression is

$$H(s) = \mathbf{c}^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + \mathtt{d} \tag{5.45}$$

We see from (5.43) that $H(s)$ will have L poles in general. However, if $\beta_j = 0$ for some j — i.e., if $\mathbf{b}$ can be expressed as a linear combination of the eigenvectors other than $\mathbf{v}_j$, see (5.32) — then λ_j fails to appear as a pole of the transfer function, even though it is still a natural frequency of the system and appears in the ZIR for almost all initial conditions. The underlying cause for this hidden mode — an internal mode that is hidden from the input/output transfer function — is evident from (5.28) or (5.41): with $\beta_j = 0$, the input fails to excite the jth mode. We say that the mode associated with λ_j is an unreachable mode in this case. In contrast, if $\beta_k \neq 0$, we refer to the kth mode as reachable. (The term controllable is also used for reachable — although strictly speaking there is a slight difference in the definitions of the two concepts in the DT case.)

If all L modes of the system are reachable, then the system itself is termed reachable, otherwise it is called unreachable. In a reachable system, the input can fully excite the state (and in fact can transfer the state vector from any specified initial condition to any desired target state in finite time). In an unreachable system, this is not possible. The notion of reachability arises in several places in systems and control theory.

The dual situation happens when $\xi_j = 0$ for some j — i.e., if $\mathbf{c}^T \mathbf{v}_j = 0$, see (5.31). In this case again, (5.43) shows that λ_j fails to appear as a pole of the transfer function, even though it is still a natural frequency of the system. Once again, we have a hidden mode. This time, the cause is evident in (5.29) or (5.42): with $\xi_j = 0$, the jth mode fails to appear at the output, even when it is present in the state response. We say that the mode associated with λ_j is unobservable in this case. In contrast, if $\xi_k \neq 0$, then we call the kth mode observable.

If all L modes of the system are observable, the system itself is termed observable, otherwise it is called unobservable. In an observable system, the behavior of the state vector can be unambiguously inferred from measurements of the input and output over some interval of time, whereas this is not possible for an unobservable system. The concept of observability also arises repeatedly in systems and control theory.

Hidden modes can cause difficulty, especially if they are unstable. However, if all we are concerned about is representing a transfer function, or equivalently the input–output relation of an LTI system, then hidden modes may be of no significance. We can obtain a reduced-order state-space model that has the same transfer function by simply discarding all the equations in (5.28) that correspond to unreachable or unobservable modes, and discarding the corresponding terms in (5.29).

The converse also turns out to be true: if a state-space model is reachable and observable, then there is no lower order state-space system that has the same transfer function; in other words, a state-space model that is reachable and observable is minimal.

Again, an entirely parallel development can be carried out for the DT case, as the next example illustrates.

EXAMPLE 5.1 A discrete-time non-minimal system

In this example we consider the DT system represented by the state equations

$$\begin{pmatrix} q_1[n+1] \\ q_2[n+1] \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & \frac{5}{2} \end{bmatrix}}_{\mathbf{A}} \begin{pmatrix} q_1[n] \\ q_2[n] \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{b}} x[n] \tag{5.46}$$

$$y[n] = \underbrace{\begin{pmatrix} -1 & \frac{1}{2} \end{pmatrix}}_{\mathbf{c}^T} \begin{pmatrix} q_1[n] \\ q_2[n] \end{pmatrix} + x[n] \tag{5.47}$$

A delay-adder-gain block diagram representing (5.46) and (5.47) is shown in Figure 5.1 below.

The modes of the system correspond to the roots of the characteristic polynomial given by

$$\det\,(\lambda \mathbf{I} - \mathbf{A}) = \lambda^2 - \frac{5}{2}\lambda + 1\ . \tag{5.48}$$

These roots are therefore

$$\lambda_1 = 2\ , \qquad \lambda_2 = \frac{1}{2}\ . \tag{5.49}$$

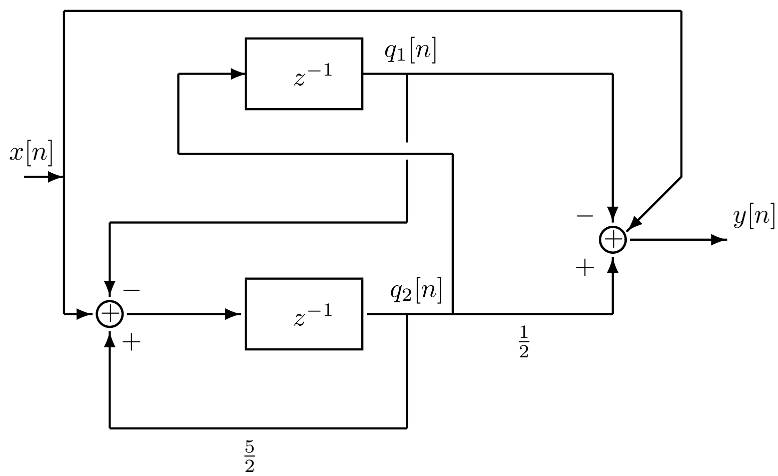

FIGURE 5.1 Delay-adder-gain block diagram for the system in Example 5.1, equations (5.46) and (5.47).

Since it is not the case here that both eigenvalues have magnitude strictly less than 1, the system is not asymptotically stable. The corresponding eigenvectors are found by solving

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda - \frac{5}{2} \end{pmatrix} \mathbf{v} = \mathbf{0} \tag{5.50}$$

with $\lambda = \lambda_1 = 2$, and then again with $\lambda = \lambda_2 = \frac{1}{2}$. This yields

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{5.51}$$

The input-output transfer function of the system is given by

$$H(z) = \mathbf{c}^T(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathtt{d} \tag{5.52}$$

$$(z\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{z^2 - \frac{5}{2}z + 1} \begin{bmatrix} z - \frac{5}{2} & 1 \\ -1 & z \end{bmatrix} \tag{5.53}$$

$$\begin{aligned} H(z) &= \frac{1}{z^2 - \frac{5}{2}z + 1} \left\{ \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} z - \frac{5}{2} & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} + 1 \\ &= \frac{1}{2} \quad \frac{z - 2}{z^2 - \frac{5}{2}z + 1} + 1 = \frac{1}{2} \quad \frac{1}{z - \frac{1}{2}} + 1 \\ &= \frac{1}{1 - \frac{1}{2}z^{-1}} \end{aligned} \tag{5.54}$$

Since the transfer function has only one pole and this pole is inside the unit circle, the system is input-output stable. However, the system has two modes, so one of them is a hidden mode, i.e., does not appear in the input-output transfer function. Hidden modes are either unreachable from the input or unobservable in the output, or both. To explicitly check which is the case in this example, we change to modal coordinates, so the original description

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \tag{5.55}$$

$$y[n] = \mathbf{c}^T\mathbf{q}[n] + \mathtt{d}x[n] \tag{5.56}$$

gets transformed via

$$\mathbf{q}[n] = \mathbf{V}\mathbf{r}[n] \tag{5.57}$$

to the form

$$\mathbf{r}[n+1] = \underbrace{\mathbf{V}^{-1}\mathbf{A}\mathbf{V}}_{\widehat{\mathbf{A}}=\mathbf{\Lambda}}\,\mathbf{r}[n] \quad + \quad \underbrace{\mathbf{V}^{-1}\mathbf{b}}_{\widehat{\mathbf{b}}=\beta}\,x[n] \tag{5.58}$$

$$y[n] = \underbrace{\mathbf{c}^T\mathbf{V}}_{\widehat{\mathbf{c}}=\xi} \quad \mathbf{r}[n] \quad + \quad \mathtt{d}x[n] \tag{5.59}$$

where

$$\mathbf{V} = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} . \tag{5.60}$$

The new state evolution matrix $\widehat{\mathbf{A}}$ will then be diagonal:

$$\widehat{\mathbf{A}} = \mathbf{\Lambda} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{5.61}$$

and the modified $\mathbf{b}$ and $\mathbf{c}$ matrices will be

$$\widehat{\mathbf{b}} = \beta = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} , \tag{5.62}$$

$$\widehat{\mathbf{c}}^T = \xi = \begin{bmatrix} 0 & -\dfrac{3}{2} \end{bmatrix} , \qquad \mathtt{d} = 1 , \tag{5.63}$$

from which it is clear that the system is reachable (because β has no entries that are 0), but that its eigenvalue $\lambda_1 = 2$ is unobservable (because ξ has a 0 in the first position). Note that if we had mistakenly applied this test in the original coordinates rather than modal coordinates, we would have erroneously decided the first mode is not reachable because the first entry of $\mathbf{b}$ is 0, and that the system is observable because $\mathbf{c}^T$ has no nonzero entries.

In the new coordinates the state equations are

$$\begin{pmatrix} r_1[n+1] \\ r_2[n+1] \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} r_1[n] \\ r_2[n] \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} x[n] \tag{5.64}$$

$$y[n] = \begin{pmatrix} 0 & -\dfrac{3}{2} \end{pmatrix} \begin{pmatrix} r_1[n] \\ r_2[n] \end{pmatrix} + x[n] \tag{5.65}$$

or equivalently

$$r_1[n+1] = 2r_1[n] + \frac{2}{3}x[n] \tag{5.66}$$

$$r_2[n+1] = \frac{1}{2}r_2[n] - \frac{1}{3}x[n] \tag{5.67}$$

$$y[n] = -\frac{3}{2}r_2[n] + x[n] \tag{5.68}$$

The delay-adder-gain block diagram represented by (5.64) and (5.65) is shown in Figure 5.2.

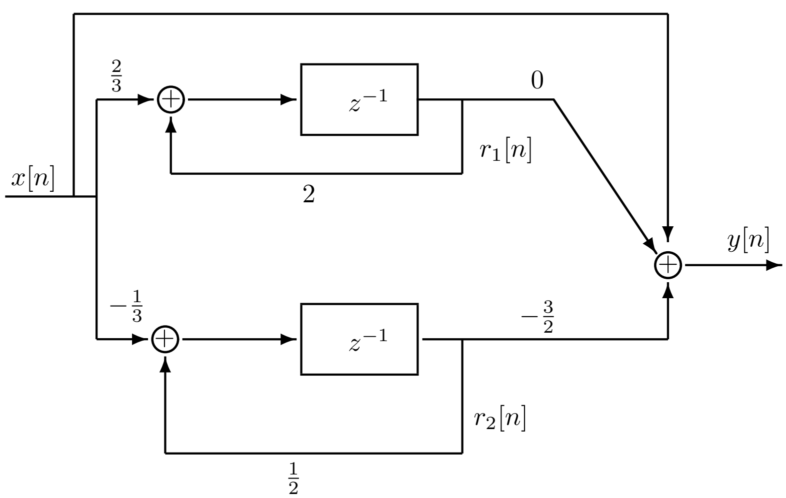

FIGURE 5.2 Delay-adder-gain block diagram for Example 5.1 after a coordinate transformation to display the modes.

In the block diagram of Figure 5.2 representing the state equations in modal coordinates, the modes are individually recognizable. This corresponds to the fact that the original $\mathbf{A}$ matrix has been diagonalized by the coordinate change. From this block diagram we can readily see by inspection that the unstable mode is not observable in the output, since the gain connecting that mode to the output is zero. However, it is reachable from the input.

Note that the block diagram in Figure 5.3 has the same modes and input-output transfer function as that in Figure 5.2. However, in this case the unstable mode is observable but not reachable.

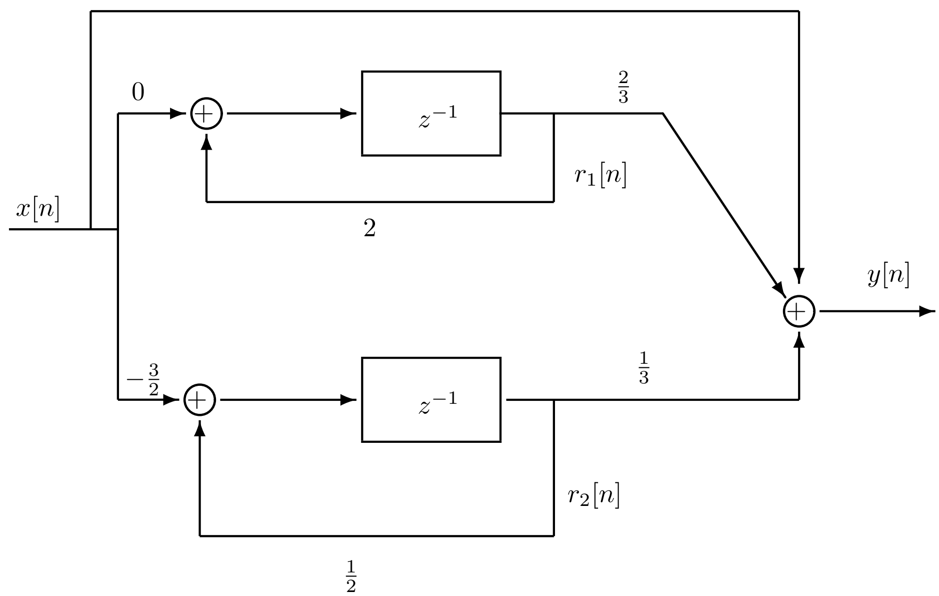

FIGURE 5.3 Delay-adder-gain block diagram for Example 5.1 realizing the same transfer function. In this case the unstable mode is observable but not reachable.

EXAMPLE 5.2 Evaluating asymptotic stability of a linear, periodically varying system

The stability of linear periodically varying systems can be analyzed by methods that are close to those used for LTI systems. Suppose, for instance, that

$$\mathbf{q}[n+1] = \mathbf{A}[n]\mathbf{q}[n] \ , \quad \mathbf{A}[n] = \mathbf{A}_0 \text{ for even } n, \ \mathbf{A}[n] = \mathbf{A}_1 \text{ for odd } n.$$

Then

$$\mathbf{q}[n+2] = \mathbf{A}_1\mathbf{A}_0\mathbf{q}[n]$$

for even n, so the dynamics of the even samples is governed by an LTI model, and the stability of the even samples is accordingly determined by the eigenvalues of the constant matrix $\mathcal{A}_{even} = \mathbf{A}_1\mathbf{A}_0$. The stability of the odd samples is similarly governed by the eigenvalues of the matrix $\mathcal{A}_{odd} = \mathbf{A}_0\mathbf{A}_1$; it turns out that the nonzero eigenvalues of this matrix are the same as those of $\mathcal{A}_{even}$, so either one can be used for a stability check.

As an example, suppose

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \ , \ \mathbf{A}_1 = \begin{pmatrix} 0 & 1 \\ 4.25 & -1.25 \end{pmatrix} \ , \tag{5.69}$$

whose respective eigenvalues are $(0\,,\,3)$ and $(1.53\,,\,-2.78)$, so both matrices have eigenvalues of magnitude greater than 1. Now

$$\mathcal{A}_{even} = A_1A_0 = \begin{pmatrix} 0 & 3 \\ 0 & 0.5 \end{pmatrix} \ , \tag{5.70}$$

and its eigenvalues are $(0\,,\,0.5)$, which corresponds to a stable system!
