Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.011: Signals, Systems and Inference

Spring 2018

Problem Set 4

Reachability and Observability, Transfer Functions, Hidden Modes, Observers

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Reading: Finish Chapter 5, and read Sections 6.1–6.2.

NOTE: Quiz 1 portions will run through the material on this problem set and everything done in lectures and recitations this coming week. You can bring 2 sheets (4 sides) of notes to the quiz.

As usual, start on this problem set early!

Problem 4.1

Problem 5.3 (which is Problem 5.13 in softcover), but instead of the forward Euler approximation given in the problem, use the *backward* Euler approximation:

$$\dot{\mathbf{q}}(nT) \approx \frac{1}{T}\Big[\mathbf{q}(nT) - q(nT - T)\Big] \ .$$

Also, assume the input $x(t)$ is identically 0, or equivalently that $\mathbf{b} = \mathbf{0}$.

[For part (c), the forward Euler approximation causes T to be limited to an upper value determined by the eigenvalues. Is there such a limit in the case of the backward Euler approximation? Of course, the smaller the value of T, the better the approximation to the CT solution in either case, but part (c) is only asking about asymptotic stability of the DT system.]

Problem 4.2

Problem 5.7 (which is Problem 5.6 in softcover), and also determine the value of the product $\beta_1\xi_1$ as well as the product $\beta_2\xi_2$, where these symbols are as defined in the chapter.

Problem 4.3

Problem 5.23 (which is Problem 5.24 in softcover), but in (b) and what follows, use

$$H_2(s) = \frac{s+2}{s-2} \ ,$$

and in (c)(ii) also determine for what values of γ the system is *BIBO* stable.

Problem 4.4

Consider an **undriven 2nd-order** LTI state-space system of the form

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] \quad \text{and} \quad y[n] = \mathbf{c}^T\mathbf{q}[n] + \mu[n] \ ,$$

where $\mu[n]$ denotes an additive noise on the measured output $y[n]$. Suppose $\mathbf{A}$ has distinct eigenvalues λ_1 and λ_2, with associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, and denote $\mathbf{c}^T\mathbf{v}_1$, $\mathbf{c}^T\mathbf{v}_2$ by ξ_1, ξ_2 respectively. Assume the system is observable, i.e., ξ_1 and ξ_2 are both nonzero.

(a) We know the initial condition can be written in the form

$$\mathbf{q}[0] = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$$

for some weights α_1 and α_2. Express $\mathbf{q}[1]$ in the same form, as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, expressing the weights in terms of α_1, α_2, λ_1, λ_2.

(b) Suppose we don't know the initial condition $\mathbf{q}[0]$. Let's see how well we can infer this initial state — or equivalently infer α_1 and α_2 — from the two measurements $y[0]$ and $y[1]$. Begin by expressing each of these output values in terms of α_1, α_2, λ_1, λ_2, ξ_1, ξ_2, $\mu[0]$, $\mu[1]$, and arranging your results in the form

$$\begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \mathbf{M} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \mu[0] \\ \mu[1] \end{bmatrix} .$$

Write down $\mathbf{M}$.

(c) Assuming no measurement noise, obtain explicit expressions for α_1 and α_2 in terms of $y[0]$ and $y[1]$.

(d) If there is measurement noise, then the actual values of α_1 and α_2 will differ from the values you computed in (c) under the assumption of no measurement noise. Use your analysis above to explain how this discrepancy between the actual and estimated value behaves in the following two cases:

(i) ξ_1 or ξ_2 becomes very small;

(ii) $\lambda_1 - \lambda_2$ becomes very small.

Problem 4.5

You can turn in your solution to this in recitation on Tuesday March 13.

A model of a rotating machine driven by a piecewise-constant torque takes the state-space form

$$\begin{aligned}\mathbf{q}[k+1] &= \begin{bmatrix} q_1[k+1] \\ q_2[k+1] \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1[k] \\ q_2[k] \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} x[k] \\ &= \mathbf{A}\mathbf{q}[k] + \mathbf{b}x[k]\end{aligned}$$

where the state vector $\mathbf{q}[k]$ comprises the position $q_1[k]$ and velocity $q_2[k]$ of the rotor, sampled at time $t = kT$; $x[k]$ is the constant value of the torque in the interval $kT \leq t < kT + T$. Assume for this problem that $T = 0.5$.

Now suppose that we have a noisy measurement of the position, so the available quantity is

$$y[k] = [1 \quad 0] \begin{bmatrix} q_1[k] \\ q_2[k] \end{bmatrix} + \zeta[k] = \mathbf{c}^T\mathbf{q}[k] + \zeta[k] \ ,$$

where $\zeta[k]$ denotes the unknown noise. One way to estimate the actual position and velocity is by using an observer, which has the form

$$\widehat{\mathbf{q}}[k+1] = \mathbf{A}\widehat{\mathbf{q}}[k] + \mathbf{b}x[k] - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \left(y[k] - \mathbf{c}^T\widehat{\mathbf{q}}[k] \right) .$$

Here $\widehat{\mathbf{q}}[k]$ is our estimate of $\mathbf{q}[k]$. Let the observer error be denoted by $\widetilde{\mathbf{q}}[k] = \mathbf{q}[k] - \widehat{\mathbf{q}}[k]$.

(a) Determine the state-space equation that $\widetilde{\mathbf{q}}[k]$ satisfies; we'll refer to this as the error equation.

(b) Show that by proper choice of the observer gains ℓ_1 and ℓ_2 we can obtain arbitrary self-conjugate natural frequencies for the error equation. What choice of ℓ_1 and ℓ_2 will place the natural frequencies of the error equation at 0 and 0.8 ?

(c) For your choice of observer gains in (b), and assuming $q_1[0] = 4$, $q_2[0] = 1$, with zero input for all time (i.e., $x[k] \equiv 0$) and zero measurement noise (i.e., $\zeta[k] \equiv 0$), set $\widehat{q}_1[0] = 0$ and $\widehat{q}_2[0] = 0$, then compare plots of $\widehat{q}_1[k]$ and $\widehat{q}_2[k]$ with plots of the underlying state variables $q_1[k]$ and $q_2[k]$, for $0 \leq k \leq 20$.

Also show plots of the estimation error $\widetilde{q}_1[k]$ and $\widetilde{q}_2[k]$ over this same time window for the case of zero-mean measurement noise $\zeta[k]$ that takes the values $+0.2$ or -0.2 with equal probability at each instant, independently of the values taken at other instants.

Problem 4.6 (Optional)

For additional practice, try Problems 5.2, 5.4, 5.11, 5.12, 5.13, 5.17, 5.18 (which are Problems 5.1, 5.3, 5.17, 5.10, 5.11, 5.7, 5.8 in softcover SSI).