

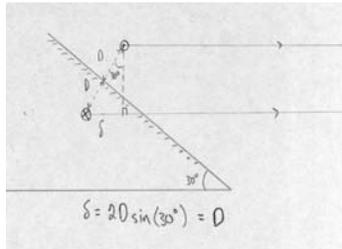
## Problem Set 11 Solutions

### Problem 11.0

The easiest way to approach this problem is to pretend that the roof antennas are transmitting to an antenna at the horizon. Then for each antenna orientation, we can find the image antenna and the distance from the roof that we need to place the antenna to maximize transmission towards the horizon.

(a)

A horizontally polarized transmitter at the horizon would require that our antenna be horizontal also (along  $\hat{y}$  from the diagram). The image of an antenna parallel to a conducting surface and height  $h$  above the surface is the same antenna half a wavelength out of phase in time a depth  $h$  below the surface. This is shown in the figure.



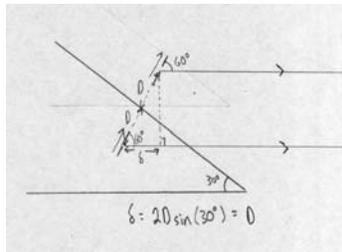
The difference in path length for the roof-top antenna and its image when looking toward the horizon will be  $D$ , the height of the antenna above the roof. In order to have a maximum toward the horizon we need the total phase delay for the image to be  $\lambda$  so we need the height  $D$  to be  $\lambda/2$ .

The antenna should be oriented along  $\hat{y}$  and placed  $D = \lambda/2$  above the roof surface.

(b)

This case is a little more difficult to solve. To receive a vertically polarized signal from the horizon (signal polarized along  $\hat{x}$ ) the antenna needs to have its axis in the x-z plane. For any antenna orientation in this plane we can break the problem into an antenna polarized perpendicular to the roof and an antenna polarized parallel to the roof (see diagram). In both cases the physical path length difference looking toward the horizon is the height of the antenna above the roof  $D$ .

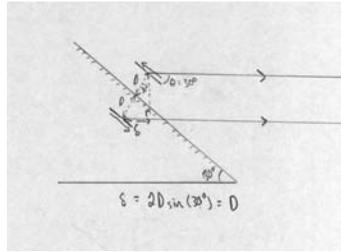
For the antenna polarized perpendicular to the roof, the antenna and its image are in phase in time so we need the path length difference to be  $\lambda$ . The total electric field in this case will be twice the field of a single antenna at an angle  $\Theta = 60^\circ$  from the dipole axis.



$$E = 2\hat{x}E_o \frac{\cos(\frac{\pi}{2} \cos(\Theta))}{\sin(\Theta)} \approx 1.63E_o.$$

For the antenna polarized parallel to the roof, the antenna and its image are out of phase by  $\lambda/2$  in time so we need the path length difference to be  $D = \lambda/2$ . The total electric field in this case will be twice the field

of a single antenna at an angle  $\Theta = 30^\circ$  from the dipole axis.



$$E = 2\hat{x}E_o \frac{\cos(\frac{\pi}{2}\cos(\Theta))}{\sin(\Theta)} \approx 0.84E_o.$$

The field for any antenna will fall between the two expressions above, so we know that the best we can do is to mount the antenna perpendicular to the roof at a height  $D = \lambda$  above the roof.

(c)

The maximum field due to a half wave dipole, using the notation above, is

$$E = \hat{x}E_o$$

The  $E_o$  will be larger in this case than for a short dipole.

The ratio of the gains is the ratio of the squares of the electric fields:

$$R = \frac{1.63^2}{1^2} = 1.63^2$$

In (dB) this is

$$R_{dB} = 10\log_{10}(R) = 20\log_{10}(1.63) = 4.24 [dB]$$

### Problem 11.1

(a)

To get the voltage induced on the receiving antenna we can use equation 10.3.19,

$$V_{TH} = -\vec{E} \cdot \vec{d}_{eff}$$

The wavelength of a 1 MHz signal is 300m, so we can say our receiving dipole is short, and the effective length is half the physical length.

Let us assume that the transmitting dipole is also a short dipole, so it will have a maximum gain of 1.5. Since we want the maximum voltage induced on the dipole lets assume that we place the receiver in the direction of maximum gain for the transmitter, and that the receiver is aligned with the electric field produced by the transmitter.

$$I = \frac{|E|^2}{2\eta_0}, \text{ so}$$

$$E = [2\eta_0 I]^{1/2}$$

We know that the intensity at the receiver is

$$I = P_t G_T(\theta, \phi) \frac{1}{4\pi r^2}, \text{ so}$$

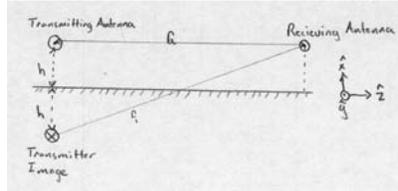
$$E = [2\eta_0 G_T(\theta, \phi) P_t \frac{1}{4\pi r^2}]^{1/2}$$

$$E = [2(377)(1.5)(1 \times 10^3) \frac{1}{4\pi} (\frac{1}{10^3})^2]^{1/2} = 0.3 \text{ [V/m]}$$

$$|V_{TH}| = |\vec{E} \cdot \vec{d}_{eff}| = 0.3 \times 0.5 = 0.15 \text{ [V]}$$

(b)

A diagram of the system is given below. Notice that the image antenna is half a wavelength out of phase (in time) with the actual antenna.



To solve this problem we need to find the total field at the receiving antenna. This is simply the sum of the field due to the antenna (ignoring the ground plane) and the field due to it's image. The general expression for the far field electric field due to a dipole of length  $d_{eff}$  driven with current  $I_0$  is:

$$\vec{E} = \hat{\Theta} j \frac{k\eta_0 I_0 d_{eff}}{4\pi r} \sin(\Theta) e^{-jkr}$$

From the diagram we know that  $\Theta = \frac{\pi}{2}$ , and that  $\hat{\Theta} = \hat{y}$  (the phase of the image will show up in the direction of the current). If the height  $h$  is much smaller than the distance  $r_a$ , we can ignore the difference between  $r_a$  and  $r_i$  in the amplitude of the waves.

$$\vec{E}_{antenna} = \hat{y} j \frac{k\eta_0 I_0 d_{eff}}{4\pi r_a} e^{-jkr_a}$$

$$\vec{E}_{image} = \hat{y} j \frac{-k\eta_0 I_0 d_{eff}}{4\pi r_i} e^{-jkr_i} \approx -\hat{y} j \frac{k\eta_0 I_0 d_{eff}}{4\pi r_a} e^{-jkr_a}$$

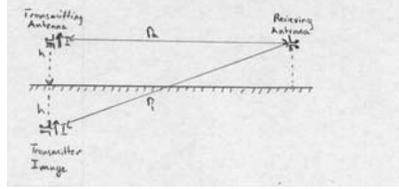
So the total field is:

$$\vec{E}_{total} = \hat{y} j \frac{k\eta_0 I_0 d_{eff}}{4\pi r_a} \{ e^{-jkr_a} - e^{-jkr_a} \} = \hat{y} j \frac{k\eta_0 I_0 d_{eff}}{4\pi r_a} e^{-jkr_a} \{ 1 - e^{-j2kh^2/r_a} \}$$

At 1-MHz,  $\lambda = 300$  [m] so if  $h \ll r_a = 1000$  [m] we can assume that  $h^2 \ll r_a \lambda$  so  $e^{-j2kh^2/r_a} \approx 1$ , and the sum is zero.

The maximum voltage in this configuration is zero.

(c)



For  $h \ll r_a$  we can say that the angle between the dipole axis and the receiving antenna is  $\frac{\pi}{2}$  for both the transmitting antenna and its image. This implies that  $\hat{\Theta} = -\hat{x}$  for both antennas. In this case, the two antennas are in phase and the fields for the two antennas are

$$\bar{E}_{antenna} = -\hat{x}j \frac{k\eta_0 I_0 d_{eff}}{4\pi_a r} e^{-jkr_a}$$

$$\bar{E}_{image} = -\hat{x}j \frac{k\eta_0 I_0 d_{eff}}{4\pi r_i} e^{-jkr_i} \approx -\hat{x}j \frac{k\eta_0 I_0 d_{eff}}{4\pi r_a} e^{-jkr_i}$$

So the total field is:

$$\bar{E}_{total} = -\hat{x}j \frac{k\eta_0 I_0 d_{eff}}{4\pi_a r} \{e^{-jkr_a} + e^{-jkr_i}\} = -\hat{x}j \frac{k\eta_0 I_0 d_{eff}}{4\pi_a r} e^{-jkr_a} \{1 + e^{-j2kh^2/r_a}\} = -\hat{x}j \frac{k\eta_0 I_0 d_{eff}}{4\pi_a r} e^{-jkr_a} 2$$

Where the last step ( $e^{-j2kh^2/r_a} \approx 1$ ) follows the same logic as in part (b).

The electric field is double what we would expect with no ground plane, so the maximum voltage is also double.

$$|V_{TH}| = 2 \times 0.15 = 0.3 \text{ [V]}$$

(c)

If we assume that a damp earth conducts fairly well, than the results of parts (b) and (c) suggest that an AM broadcast radio station operating near 1-MHz should use vertically polarized antennas.

The results from parts (b) and (c) required that  $h^2 \ll r_a \lambda$ . This was true at 1-MHz where the wavelength was 300 meters, but we don't know if this will be true for 1-GHz where the wavelength is 0.3 meters. Consider the case of  $h = 10$ [m].  $h^2 = 100 \ll 1000 \times 300$ , but  $h^2 = 100! \ll 1000 \times 0.3$ . So this argument does not apply to cell phones operating above 1-GHz.

**Problem 11.2****(a)**

We want to find the frequency separation between adjacent nulls. For the first null we know that

$kL_1 - kL_2 = k(L_1 - L_2) = n\pi$  with  $n$  odd (because the path length difference has to be an odd multiple of half a wavelength). We can re-write this in terms of frequency:

$$k(L_1 - L_2) = f \frac{2\pi}{c} (L_1 - L_2) = n\pi, \text{ or}$$

$$f = \frac{nc}{2} \frac{1}{L_1 - L_2}, \text{ with } n \text{ odd.}$$

The next null will occur when the path length difference is now  $n + 2$  times half the wavelength.

$$f_2 = \frac{(n+2)c}{2} \frac{1}{L_1 - L_2}$$

So the bandwidth is  $f_2 - f$  which is:

$$B = \frac{(n+2)c}{2} \frac{1}{L_1 - L_2} - \frac{nc}{2} \frac{1}{L_1 - L_2}$$

$$B = \frac{c}{L_1 - L_2} = \frac{3 \times 10^8}{11 \times 10^3 - 10 \times 10^3} = \frac{3 \times 10^8}{1 \times 10^3}$$

$$B = 3 \times 10^5 [Hz] = 0.3 [MHz]$$

This suggests that the 6-MHz wide TV signals will have problems with multipath propagation with path length differences in the 1 km range.

**(b)**

To give a good answer we would have to have a better idea of how many nulls we could tolerate in any one sub-band. If we assume that just one null in the sub-band would be ok, then we want the individual sub-bands to be 0.3 MHz (N=20) for this type of problem. In the best case this would put the nulls between adjacent sub-bands, and in the worst case there would be one null in each of the sub-bands.

### Problem 11.3

The easiest way to solve this problem is to solve for the total far field electric field in the variables  $\Theta$  and  $\phi$  as defined for the diagram. The easiest way to get the far field pattern is to recognize that the total field will be the product of the far field pattern for a single patch with the array factor for two antennas driven in phase but separated by  $200\lambda$  along the  $\hat{y}$  axis.

We can get the far field of a single patch antenna from equation 11.1.15 by substituting  $100\lambda$  for  $L_x$  and  $L_z$ ,  $\Theta$  for  $\alpha_x$ , and  $\phi$  for  $\alpha_z$ .

$$\bar{E}(\Theta, \phi) = \hat{\Theta} \frac{j}{\lambda r} E_o e^{-jkr} (100\lambda)(100\lambda) \left[ \frac{\sin(100\pi\Theta)}{100\pi\Theta} \right] \left[ \frac{\sin(100\pi\phi)}{100\pi\phi} \right]$$

If we group all the uninteresting terms (those that don't change with  $\Theta$  or  $\phi$ ) into one complex number  $E'_o$  we can re-write this as

$$\bar{E}(\Theta, \phi) = \hat{\Theta} E'_o \left[ \frac{\sin(100\pi\Theta)}{100\pi\Theta} \right] \left[ \frac{\sin(100\pi\phi)}{100\pi\phi} \right]$$

The array factor is

$$F = e^{jkr_1} + e^{jkr_2} = e^{jkr_1} + e^{jkr_1} e^{jk\delta} = e^{jkr_1} e^{jk\delta/2} [e^{-jk\delta/2} + e^{jk\delta/2}] = e^{jkr_1} e^{jk\delta/2} 2\cos(\frac{k}{2}\delta)$$

$$F = e^{jkr_1} e^{jk\delta/2} 2\cos(\frac{2\pi}{\lambda} 2D \sin(\phi)) = e^{jkr_1} e^{jk\delta/2} 2\cos(\frac{\pi}{\lambda} 200\lambda \sin(\phi)) = e^{jkr_1} e^{jk\delta/2} 2\cos(200\pi \sin(\phi))$$

Again we can group all the terms that do not change with  $\Theta$  or  $\phi$ ,

$$F = \underline{F}'_o \cos(200\pi \sin(\phi))$$

So the total field is

$$\bar{E}_T = \hat{\Theta} E'_o \underline{F}'_o \left[ \frac{\sin(100\pi\Theta)}{100\pi\Theta} \right] \left[ \frac{\sin(100\pi\phi)}{100\pi\phi} \right] \cos(200\pi \sin(\phi))$$

(a)

The nulls in the x-z plane occur when  $\left[ \frac{\sin(100\pi\Theta)}{100\pi\Theta} \right] = 0$ . So, the first null occurs at  $\Theta = \frac{1}{100}$ .

(b)

In the y-z plane the nulls occur when either  $\left[ \frac{\sin(100\pi\phi)}{100\pi\phi} \right] = 0$  or  $\cos(200\pi \sin(\phi)) = 0$ . The first condition is met when  $\phi = \frac{1}{100}, \frac{2}{100}, \frac{3}{100}, \dots$ . The second condition is met when  $\sin(\phi) = \frac{1}{400}, \frac{3}{400}, \frac{5}{400} \dots$ . For small values of  $\phi$ ,  $\sin(\phi) = \phi$ .

So, the first three nulls occur at  $\phi = \frac{1}{400}$ ,  $\phi = \frac{3}{400}$ , and  $\phi = \frac{5}{400}$ .

(c)

There are two approaches to the problem. The first is to evaluate the square sinc function at the first two maxima of  $|E|^2$  in the x-z plane. The second is to look at the vector addition that represents the complex addition of the phase terms corresponding to the integral over the aperture.

First lets look at the maxima of  $\text{sinc}^2(x)$ . We know the first occurs at  $x = 0$ , with value 1. To find the next maxima we could take the derivative of  $\text{sinc}^2(x)$  and set it equal to zero and then evaluate to see if we're at a zero or maxima. The easier method is to recognize that the local minima and maxima of  $\text{sinc}(x)$  are the points we're looking for.

$$\frac{d}{dx} \frac{\sin(x)}{x} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} = 0$$

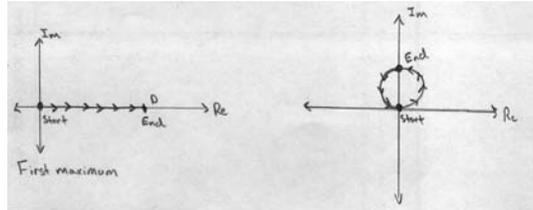
Which are the points when  $\cos(x) = \frac{\sin(x)}{x}$ . So the first maxima after  $x = 0$  is  $x = 4.4934 \text{ rad}$ .

So the ratio of gains is  $\frac{0.2172^2}{1^2} = 0.0472$

To solve this problem graphically, see the attached diagram.

The first maxima occurs when all the vectors have zero imaginary part, and is proportional to  $D$ . The first zero occurs when the vector sum starts and ends at the same point (forms a circle with circumference  $D$ ). The next maxima occurs when the vector sum forms a circle and ends halfway around the circle from the start. In this case the circumference of the circle is  $C = D/3 = \pi A$ , so  $A$  (which is proportional to the magnitude of the sinc function) is  $A = D \frac{2}{3\pi}$ .

The ratio is then  $\frac{D^2 4 / (9\pi^2)}{D^2} = \frac{4}{9\pi^2} = 0.045$



**Problem 11.4**

(a)

$$P_{TH} = k_B T_R B = (1.38 \times 10^{-23})(100)(5 \times 10^3) = 6.9 \times 10^{-18} [W]$$

(b)

$$P_{signal} = 10 \times P_{TH} = 10(1.38 \times 10^{-23})(100)(5 \times 10^3) = I_R A_{eff} = 6.9 \times 10^{-17} [W]$$

$$I_R = P_t G_T \frac{1}{4\pi r^2} = (1)(1)(4\pi)^{-1}(3 \times 10^5)^{-2} = 8.84 \times 10^{-13} [W/m^2]$$

$$A_{eff} = 10 \times P_{TH} / I_R = 10(1.38 \times 10^{-23})(100)(5 \times 10^3)(4\pi)(3 \times 10^5)^2 = 7.8 \times 10^{-5} [m^2]$$

So the required antenna effective area is  $7.8 \times 10^{-5} [m^2]$ .

Note that the actual distance is closer to  $3 \times 10^5$  km, which gives us an area of 78  $[m^2]$

(c)

The effective area of an aperture antenna is approximately the physical area of the aperture. We can assume that the antenna is square, then the length of each side is  $L_x = L_y = \sqrt{A_{eff}}$ .

From equation 11.1.26 we know that the first null will occur at an angle

$$\Theta_n = \pm.$$

So the spot size on the ground will be

$$W = 2r \tan(\Theta_n) = 2r \tan(\text{asin}(\lambda / \sqrt{A_{eff}}))$$

For the distance  $3 \times 10^5$  [m] this becomes:

$$W = 2(3 \times 10^5) \tan(\text{asin}(0.003/0.00883)) = 2.002 \times 10^5 [m] \text{ between the first nulls. The area associated with this is } Spot \text{ Area} = W^2 = 4.008 \times 10^{10} [m^2]$$

For the distance  $3 \times 10^5$  [km] this becomes:

$$W = 2(3 \times 10^8) \tan(\text{asin}(0.003/8.83)) = 2.04 \times 10^5 [m] \text{ between the first nulls. The area associated with this is } Spot \text{ Area} = W^2 = 4.15 \times 10^{10} [m^2]$$

(d)

If the spot size is smaller than the diameter of the earth (which we're told to assume) then we can say that the field of view of the antenna is filled with the earth at a brightness temperature of  $T_B = 250K$ . We can use the result of section 11.3.3 to say that the antenna temperature is now  $T_R = T_B = 250K$ . This is 2.5 times the previous value, so our effective area (which grows linearly with the receiver thermal noise power) will be 2.5 times its previous value.

$$A_{eff, new} = 2.5 \times A_{eff, old}$$

$$\text{For the distance } 3 \times 10^5 [m] \text{ this becomes } A = 1.95 \times 10^{-4} [m^2]$$

$$\text{For the distance } 3 \times 10^5 [km] \text{ this becomes } A = 195 [m^2]$$

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