6.01: Introduction to EECS I

Characterizing System Performance

March 1, 2011

Midterm Examination #1

Time: Tuesday, March 8, 7:30 PM to 9:30 PM

Location: Walker Memorial (if last name starts with A-M)
10-250 (if last name starts with N-Z)

Coverage: Everything up to and including Design Lab 5.

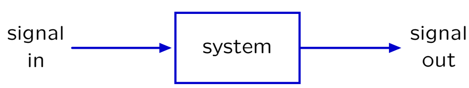
You may refer to any printed materials that you bring to exam.

You may not use computers or phones.

No software lab in week 6.

The Signals and Systems Abstraction

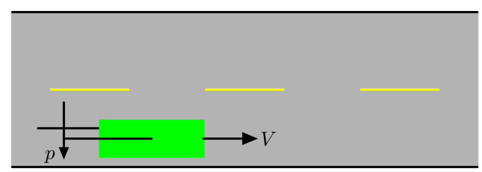
Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



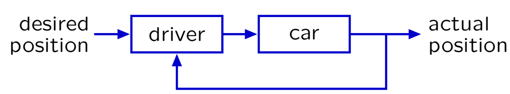
Today's goals: use signals and systems approach
– to gain insight into how feedback works.
– to characterize the responses of systems quantitatively.

Feedback and Control

Feedback is pervasive in natural and artificial systems.

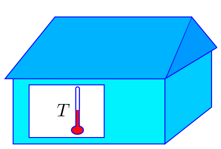


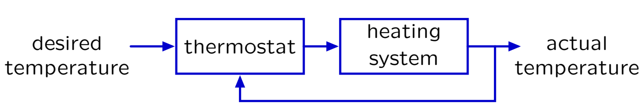
Turn steering wheel to stay centered in the lane.



Feedback and Control

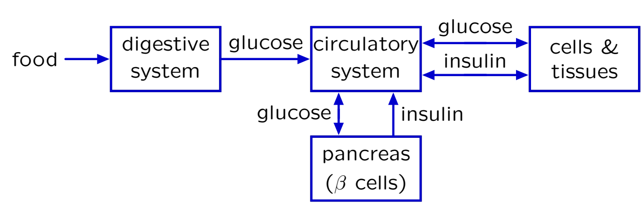
Feedback is useful for regulating a system's behavior, as when a thermostat regulates the temperature of a house.

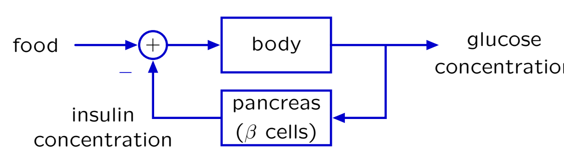




Feedback and Control

Concentration of glucose in blood is highly regulated and remains nearly constant despite episodic ingestion and use.





Feedback and Control

Motor control relies on feedback from pressure sensors in the skin as well as proprioceptors in muscles, tendons, and joints.

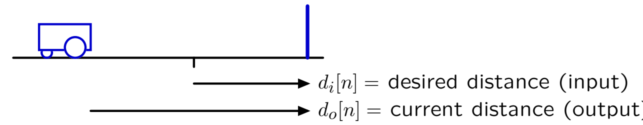
Try building a robotic hand to unscrew a lightbulb:

Shadow Dexterous Robot Hand
(Greenhill and Elias)

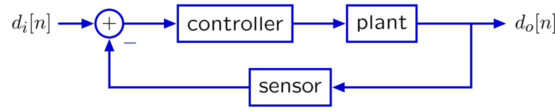
... wouldn't be possible without feedback!

Example from Design Lab 4: WallFinder

Control the robot to move to desired distance from a wall.

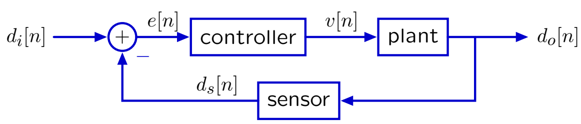


Think about this system as having 3 parts:



Example from Design Lab 4: WallFinder

Control the robot to move to desired distance from a wall.



Controller (brain) – sets velocity $\propto$ error:

$$v[n] = ke[n] = k\big(d_i[n] - d_s[n]\big)$$

Plant (robot locomotion) – given $v[n]$, drives to new position:

$$d_o[n] = d_o[n-1] - Tv[n-1]$$

Sensor (sonars) – reports output, but also introduces delay:

$$d_s[n] = d_o[n-1]$$

Check Yourself

Solving difference equations.

$$v[n] = k\big(d_i[n] - d_s[n]\big)$$
$$d_o[n] = d_o[n-1] - Tv[n-1]$$
$$d_s[n] = d_o[n-1]$$

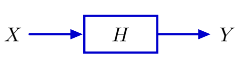
How many equations? How many unknowns?

1. 3 equations; 3 unknowns
2. 3 equations; 4 unknowns
3. 4 equations; 4 unknowns
4. 3 equations; 6 unknowns
5. none of the above

Hint: k and T are fixed (constant) parameters.

System Functional

We can express the relation between the (known) input and (unknown) output using the system functional H.



The system functional H is an **operator**.
Applying H to X yields Y.

$$Y = HX$$

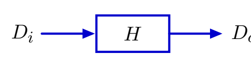
It is also convenient to think of H as a ratio:

$$H = \frac{Y}{X}$$

because H can be expressed as a ratio of polynomials in $\mathcal{R}$.

Check Yourself

Determine the system functional for the WallFinder system.



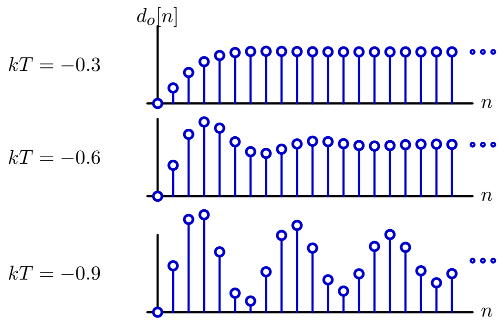
$$V = kE = k\big(D_i - D_s\big)$$
$$D_o = \mathcal{R}D_o - \mathcal{R}TV$$
$$D_s = \mathcal{R}D_o$$

Find the system functional $H = \dfrac{D_o}{D_i}$.

1. $\dfrac{kT\mathcal{R}}{1-\mathcal{R}}$
2. $\dfrac{-kT\mathcal{R}}{1+\mathcal{R}+kT\mathcal{R}^2}$
3. $\dfrac{-kT\mathcal{R}}{1-\mathcal{R}-kT\mathcal{R}^2}$
4. $\dfrac{-kT\mathcal{R}}{1-\mathcal{R}} + kT$
5. none of above

Example from Design Lab 4: WallFinder

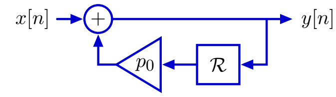
We saw in lab that the behavior of WallFinder depends on kT.



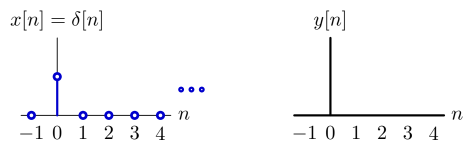
We would like to relate these behaviors to the system functional H, so that we can design well-behaved control systems.

Feedback

Consider a simpler system with **feedback**.

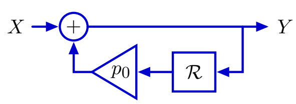


Find $y[n]$ given $x[n] = \delta[n]$: $y[n] = x[n] + p_0 y[n-1]$



Feedback

Alternatively, we can think about signals instead of samples.



$Y = X + p_0\mathcal{R}Y$

$(1 - p_0\mathcal{R})Y = X$

$$H = \frac{Y}{X} = \frac{1}{1 - p_0\mathcal{R}}$$

Feedback

The reciprocal of $1 - p_0\mathcal{R}$ can be evaluated using synthetic division.

$$\begin{array}{r|l}
 & 1 \;\; +p_0\mathcal{R} \;\; +p_0^2\mathcal{R}^2 \;\; +p_0^3\mathcal{R}^3 \;\; +\cdots \\
1 - p_0\mathcal{R} & 1 \\
 & 1 \;\; -p_0\mathcal{R} \\
 & \quad p_0\mathcal{R} \\
 & \quad p_0\mathcal{R} \;\; -p_0^2\mathcal{R}^2 \\
 & \qquad p_0^2\mathcal{R}^2 \\
 & \qquad p_0^2\mathcal{R}^2 \;\; -p_0^3\mathcal{R}^3 \\
 & \qquad\quad p_0^3\mathcal{R}^3 \\
 & \qquad\quad p_0^3\mathcal{R}^3 \;\; -p_0^4\mathcal{R}^4 \\
 & \qquad\qquad \cdots
\end{array}$$

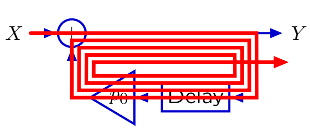
Therefore

$$\frac{1}{1 - p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots$$

Feedback: Cyclic Signal Flow Paths

Feedback implies cyclic signal flow paths.

$x[n] = \delta[n]$

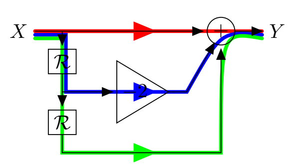


$$\frac{Y}{X} = \frac{1}{1 - p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots$$

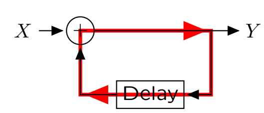
Cyclic signal flow paths → **persistent responses** to transient inputs.

Cyclic and Acyclic Signal Flow Paths

Block diagrams help visualize feedback as cyclic signal flow paths.



acyclic



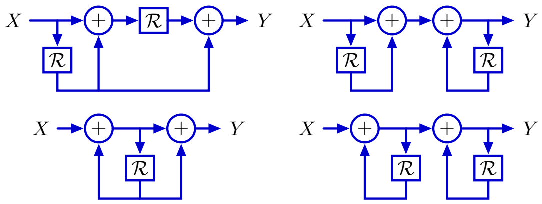
cyclic

Acyclic: no cycles in any path from input to output

Cyclic: at least one cycle

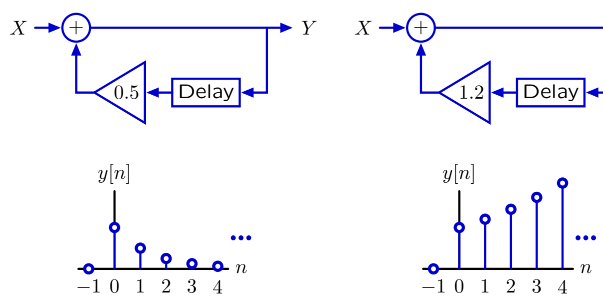
Check Yourself

How many of the following systems are cyclic?



Geometric Growth

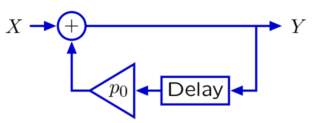
If traversing the cycle decreases or increases the magnitude of the signal, then the output will decay or grow, respectively.



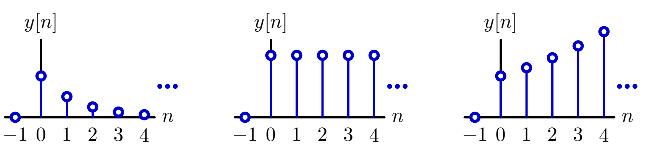
→ geometric sequences: $y[n] = (0.5)^n$ and $(1.2)^n$ for $n \geq 0$.

Geometric Growth

These system responses can be characterized by a single number (the **pole**), which is the base of the geometric sequence.



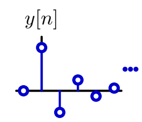
$$y[n] = \begin{cases} p_0^n, & \text{if } n >= 0; \\ 0, & \text{otherwise.} \end{cases}$$



$p_0 = 0.5$ $p_0 = 1$ $p_0 = 1.2$

Check Yourself

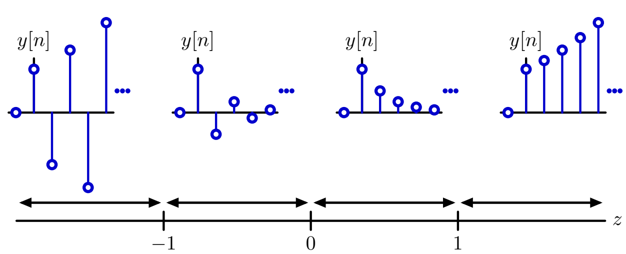
What value of p_0 represents the signal below?



1. $p_0 = 0.5$
2. $p_0 = -0.5$
3. $p_0 = 0.25$ interspersed with $p_0 = -0.25$
4. none of the above

Geometric Growth

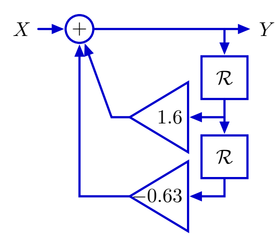
The value of p_0 determines the rate of growth.



$p_0 > 1$: magnitude diverges monotonically
$0 < p_0 < 1$: magnitude converges monotonically
$-1 < p_0 < 0$: magnitude converges, alternating sign
$p_0 < -1$: magnitude diverges, alternating sign

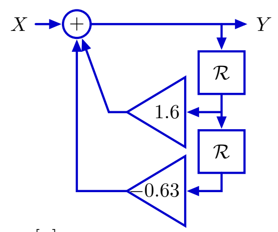
Second-Order Systems

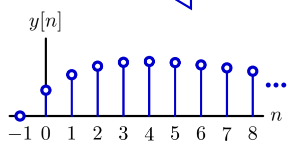
The unit-sample response of more complicated cyclic systems is more complicated.



Second-Order Systems

The unit-sample response of more complicated cyclic systems is more complicated.

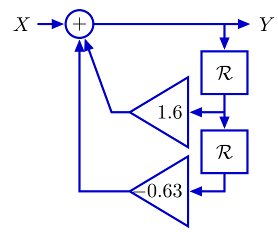




Not geometric. This response grows then decays.

Second-Order Systems: Equivalent forms

Factor the operator expression to break the system into two simpler systems (divide and conquer).



$$Y = X + 1.6\mathcal{R}Y - 0.63\mathcal{R}^2 Y$$

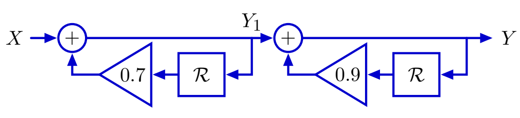
$$(1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2)\,Y = X$$

$$(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})\,Y = X$$

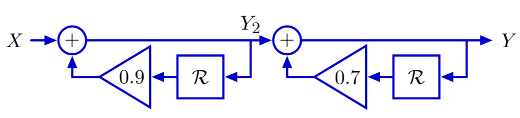
Second-Order Systems: Equivalent forms

Factored form corresponds to a cascade of simpler systems.

$$(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})\,Y = X$$



$(1 - 0.7\mathcal{R})\,Y_1 = X$ $\quad$ $(1 - 0.9\mathcal{R})\,Y = Y_1$

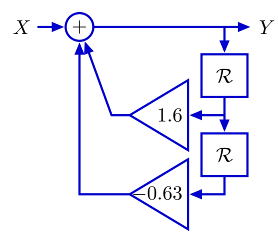


$(1 - 0.9\mathcal{R})\,Y_2 = X$ $\quad$ $(1 - 0.7\mathcal{R})\,Y = Y_2$

The order doesn't matter (if systems initially at rest).

Second-Order Systems: Equivalent forms

This system functional can also be written as a **sum** of simpler parts.



$$Y = X + 1.6\mathcal{R}Y - 0.63\mathcal{R}^2 Y$$

$$(1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2)\,Y = X$$

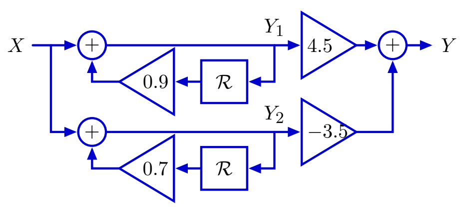
$$(1 - 0.9\mathcal{R})(1 - 0.7\mathcal{R})\,Y = X$$

$$\frac{Y}{X} = \frac{1}{1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2} = \frac{1}{(1 - 0.9\mathcal{R})(1 - 0.7\mathcal{R})} = \frac{4.5}{1 - 0.9\mathcal{R}} - \frac{3.5}{1 - 0.7\mathcal{R}}$$

Second-Order Systems: Equivalent forms

The unit-sample response is the **sum of geometric sequences**.

$$\frac{Y}{X} = \frac{4.5}{1 - 0.9\mathcal{R}} - \frac{3.5}{1 - 0.7\mathcal{R}}$$



If $x[n] = \delta[n]$ then $y_1[n] = 0.9^n$ and $y_2[n] = 0.7^n$ for $n \ge 0$.

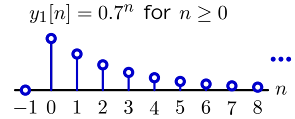
Thus, $y[n] = 4.5(0.9)^n - 3.5(0.7)^n$ for $n \ge 0$.

It would be difficult (or impossible) to derive this representation directly from the original block diagram or difference equation.

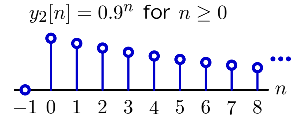
Second-Order Systems: Equivalent forms

The unit-sample response is the **sum of geometric sequences**.

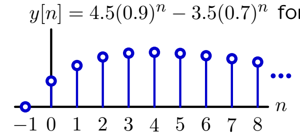
$y_1[n] = 0.7^n$ for $n \ge 0$



$y_2[n] = 0.9^n$ for $n \ge 0$



$y[n] = 4.5(0.9)^n - 3.5(0.7)^n$ for $n \ge 0$



Higher-Order Systems

Systems that can be represented by linear difference equations with constant coefficients have operator representations that are the ratio of polynomials in $\mathcal{R}$.

$$y[n]+a_1y[n-1]+a_2y[n-2]+a_3y[n-3]+\cdots$$
$$=b_0x[n]+b_1x[n-1]+b_2x[n-2]+b_3x[n-3]+\cdots$$

$$(1+a_1\mathcal{R}+a_2\mathcal{R}^2+a_3\mathcal{R}^3+\cdots)\,Y=(b_0+b_1\mathcal{R}+b_2\mathcal{R}^2+b_3\mathcal{R}^3+\cdots)\,X$$

$$\frac{Y}{X}=\frac{b_0+b_1\mathcal{R}+b_2\mathcal{R}^2+b_3\mathcal{R}^3+\cdots}{1+a_1\mathcal{R}+a_2\mathcal{R}^2+a_3\mathcal{R}^3+\cdots}$$

Rational Polynomial: ratio of two polynomials

Poles

Poles can be identified by expanding the system functional in partial fractions.

$$\frac{Y}{X}=\frac{b_0+b_1\mathcal{R}+b_2\mathcal{R}^2+b_3\mathcal{R}^3+\cdots}{1+a_1\mathcal{R}+a_2\mathcal{R}^2+a_3\mathcal{R}^3+\cdots}$$

Factor denominator:

$$\frac{Y}{X}=\frac{b_0+b_1\mathcal{R}+b_2\mathcal{R}^2+b_3\mathcal{R}^3+\cdots}{(1-p_0\mathcal{R})(1-p_1\mathcal{R})(1-p_2\mathcal{R})(1-p_3\mathcal{R})\cdots}$$

Partial fractions:

$$\frac{Y}{X}=\frac{e_0}{1-p_0\mathcal{R}}+\frac{e_1}{1-p_1\mathcal{R}}+\frac{e_2}{1-p_2\mathcal{R}}+\cdots+f_0+f_1\mathcal{R}+f_2\mathcal{R}^2+\cdots$$

The poles are p_i for $0\le i<n$ where n is the order of the denominator.

One geometric mode p_i^n arises from each factor of the denominator.

Poles

Alternatively, replace each $\mathcal{R}$ in the system functional by $1/z$.
Then the poles are the roots of the denominator polynomial in z.

Start with system functional:

$$\frac{Y}{X}=\frac{1}{1-1.6\mathcal{R}+0.63\mathcal{R}^2}=\frac{1}{(1-p_0\mathcal{R})(1-p_1\mathcal{R})}=\frac{1}{\underbrace{(1-0.7\mathcal{R})}_{p_0=0.7}\underbrace{(1-0.9\mathcal{R})}_{p_1=0.9}}$$

Substitute $1/z$ for $\mathcal{R}$ and find roots of denominator polynomial in z:

$$\frac{Y}{X}=\frac{1}{1-\frac{1.6}{z}+\frac{0.63}{z^2}}=\frac{z^2}{z^2-1.6z+0.63}=\frac{z^2}{\underbrace{(z-0.7)}_{z_0=0.7}\underbrace{(z-0.9)}_{z_1=0.9}}$$

Poles are at 0.7 and 0.9

Check Yourself

Consider the system described by

$$y[n]=-\frac{1}{4}y[n-1]+\frac{1}{8}y[n-2]+x[n-1]-\frac{1}{2}x[n-2]$$

How many of the following are true?

1. The unit sample response converges to zero.
2. There are poles at $z=\frac{1}{2}$ and $z=\frac{1}{4}$.
3. There is a pole at $z=\frac{1}{2}$.
4. There are two poles.
5. None of the above

Complex Poles

What if a pole has a non-zero imaginary part?

Example:

$$\frac{Y}{X}=\frac{1}{1-\mathcal{R}+\mathcal{R}^2}$$
$$=\frac{1}{1-\frac{1}{z}+\frac{1}{z^2}}=\frac{z^2}{z^2-z+1}$$

Poles are $z=\frac{1}{2}\pm\frac{\sqrt{3}}{2}j=e^{\pm j\pi/3}$.

What are the implications of complex poles?

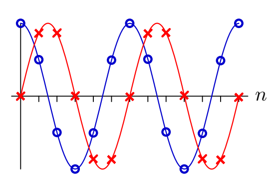
Complex Poles

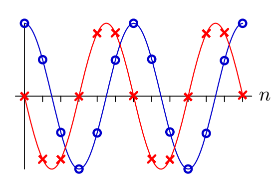
Partial fractions work even when the poles are complex.

$$\frac{Y}{X}=\frac{1}{1-e^{j\pi/3}\mathcal{R}}\times\frac{1}{1-e^{-j\pi/3}\mathcal{R}}=\frac{1}{j\sqrt{3}}\left(\frac{e^{j\pi/3}}{1-e^{j\pi/3}\mathcal{R}}-\frac{e^{-j\pi/3}}{1-e^{-j\pi/3}\mathcal{R}}\right)$$

There are two fundamental modes (both geometric sequences):

$e^{jn\pi/3}=\cos(n\pi/3)+j\sin(n\pi/3)$ and $e^{-jn\pi/3}=\cos(n\pi/3)-j\sin(n\pi/3)$

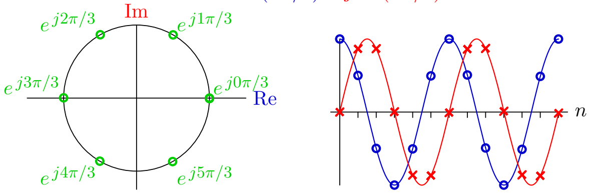




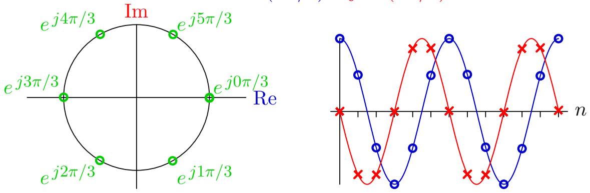
Complex Poles

Complex modes are easier to visualize in the complex plane.

$e^{jn\pi/3} = \cos(n\pi/3) + j\sin(n\pi/3)$



$e^{-jn\pi/3} = \cos(n\pi/3) - j\sin(n\pi/3)$



Complex Roots

Difference equations that represent physical systems (e.g., population growth, bank accounts, etc.) have real-valued coefficients.

Bank account with interest:

$$y[n] = (1+r)y[n-1] + x[n]$$

wallFinder:

$$d_o[n] = d_o[n-1] + kTd_o[n-2] - kTd_i[n-1]$$

Difference equations with real-valued coefficients generate real-valued outputs from real-valued inputs.

Complex Roots

If p is a root of a polynomial with constant real-valued coefficients, then its complex-conjugate p^* is also a root.

Proof. Let $D(z)$ represent a polynomial in z with constant real-valued coefficients.

If p is a root of $D(z)$ then $D(p) = 0$.

Since all of the coefficients are real-valued,

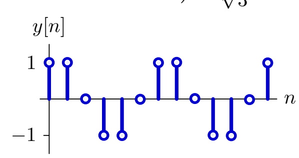
$$D(p^*) = (D(p))^* = 0^* = 0.$$

Thus p^* is also a root.

Complex Roots

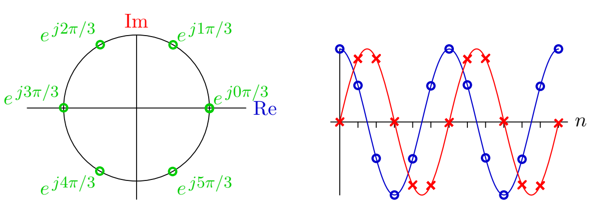
If we pair the factors corresponding to complex-conjugate roots, the resulting polynomial has real-valued coefficients.

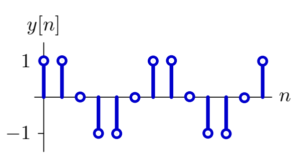
$$H = \frac{Y}{X} = \frac{1}{1-\mathcal{R}+\mathcal{R}^2} = \frac{1}{1-e^{j\pi/3}\mathcal{R}} \times \frac{1}{1-e^{-j\pi/3}\mathcal{R}}$$
$$= \frac{1}{j\sqrt{3}}\left(\frac{e^{j\pi/3}}{1-e^{j\pi/3}\mathcal{R}} - \frac{e^{-j\pi/3}}{1-e^{-j\pi/3}\mathcal{R}}\right)$$
$$y[n] = \frac{1}{j\sqrt{3}}\left(e^{j(n+1)\pi/3} - e^{-j(n+1)\pi/3}\right) = \frac{2}{\sqrt{3}}\sin\frac{(n+1)\pi}{3}$$



Complex Roots

Furthermore, the period of the resulting real-valued signal is the same as the periods of the complex-valued signals.

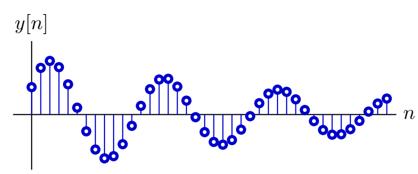




Thus the period of the response is equal to the number of steps required to go around the unit circle (here 6).

Check Yourself

Output of a system with poles at $z = re^{\pm j\Omega}$.



Which statement is true?

1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

Fibonacci's Bunnies

We can think about the Fibonacci numbers as the output of a discrete-time system.

'How many pairs of rabbits can be produced from a single pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?'

Difference equation model:

$$y[n] = x[n] + y[n-1] + y[n-2]$$

What does the input $x[n]$ represent?

Fibonacci's Bunnies

We can think about the Fibonacci numbers as the output of a discrete-time system.

'How many pairs of rabbits can be produced from a single pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?'

Difference equation model:

$$y[n] = x[n] + y[n-1] + y[n-2]$$

If the system is started "at rest" and $x[n] = \delta[n]$, then

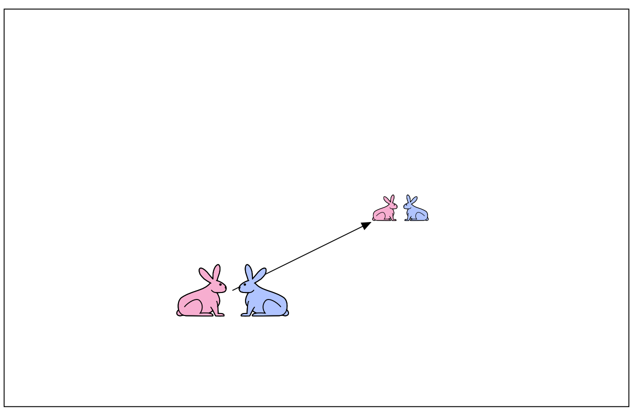
$y[0] = 1$

$y[1] = 1$

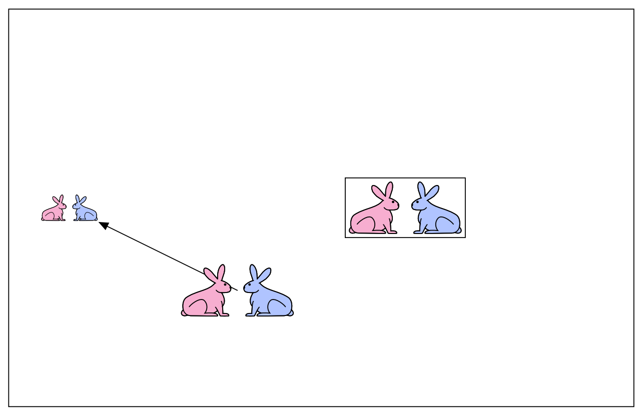
$y[2] = 2$

$y[3] = 3$

Fibonacci's Bunnies



Fibonacci's Bunnies



Check Yourself

What are the poles of the Fibonacci system?

$$y[n] = x[n] + y[n-1] + y[n-2]$$

Summary

Feedback → cyclic signal flow paths.

Cyclic signal flow paths → persistent responses to transient inputs.

We can characterize persistent responses with poles.

Poles provide a way to characterize the behavior of a system in terms of a mathematical description as a system function.

Summary

Feedback → cyclic signal flow paths.

Cyclic signal flow paths → persistent responses to transient inputs.

We can characterize persistent responses with poles.

Poles provide a way to characterize the behavior of a system in terms of a mathematical description as a system function.

Powerful representations (here polynomials)
→ powerful abstractions (e.g., **poles**)

PCAP !