Tutorial 08
April 13-14, 2006

1. For this problem we will need to compute the 1st, 2nd, 3rd, and 4th moments of the standard normal distribution. To facilitate this, we will find the moment generating function:

$$\begin{aligned} E[e^{rx}] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{rx}\cdot e^{-\frac{1}{2}x^2}dx \\ &= \frac{1}{\sqrt{2\pi}}\cdot e^{\frac{1}{2}r^2}\cdot\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-r)^2}dx \\ &= e^{\frac{1}{2}r^2} \end{aligned}$$

the second equality above following from completing the square in the exponent, and the third following because the gaussian density function integrates to 1. Therefore we now take derivatives w.r.t. r, and easily find the first four moments:

$$E[X]=0, E[X^2]=1, E[X^3]=0, E[X^4]=3.$$

We know that the correlation coefficient is given by:

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}.$$

We first compute the covariance:

$$\begin{aligned} Cov(X,Y) &= E[XY]-E[X]E[Y] \\ &= E[aX+bX^2+cX^3]-E[X]E[Y] \\ &= aE[X]+bE[X^2]+cE[X^3] \\ &= b. \end{aligned}$$

Now $Var(X)=1$ therefore $\sigma_X=1$ so we have left to find $\sigma_Y=\sqrt{Var(Y)}$.

$$\begin{aligned} Var(Y) &= Var(a+bX+cX^2) \\ &= E[(a+bX+cX^2)^2]-E[a+bX+cX^2]^2 \\ &= (a^2+2ac+b^2+3c^2)-(a^2+c^2+2ac) \\ &= b^2+2c^2 \end{aligned}$$

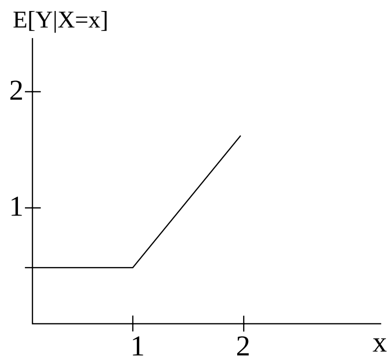
and therefore we find:

$$\rho(X,Y)=\frac{b}{\sqrt{b^2+2c^2}}.$$

2. (a) Here we are trying to choose a $g(X)$ that minimizes the conditional mean squared error $E[(Y-g(X))^2|X=x]$. As shown in section 4.6 in the text, this estimator is $g(X)=E[Y|X=x]$.

$$E[Y|X=x] = \begin{cases} \frac{1}{2} & 0\le x<1 \\ x-\frac{1}{2} & 1\le x\le 2 \\ Undefined & Otherwise \end{cases}$$

A plot of $g(X)$:



(b) $E[g^*(X)] = E[E[Y|X = x]] = E[Y]$, and note that $f_{X,Y}(x, y) = C = \frac{1}{2}$.

$$\begin{aligned} E[Y] &= \int_x \int_y y f_{X,Y}(x,y) dy dx \\ &= \int_0^1 \int_0^1 y \frac{1}{2} dy dx + \int_1^2 \int_{x-1}^{x} y \frac{1}{2} dy dx \\ &= \int_0^1 \frac{1}{4} dx + \int_1^2 \frac{1}{4}(2x-1)\, dx \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(g^*(X)) &= \mathrm{Var}(E[Y|X = x]) = E[E[Y|X]^2] - (E[E[Y|X]])^2 \\ &= \int_0^2 E[Y|X]^2 f_X(x) dx - (E[Y])^2 \\ &= \int_0^1 (\frac{1}{2})^2 \frac{1}{2} dx + \int_1^2 (x - \frac{1}{2})^2 \frac{1}{2} dx - (\frac{3}{4})^2 \\ &= \frac{5}{48} \end{aligned}$$

, where we have noted that $f_X(x) = \int_y f_{X,Y}(x, y)dy = \frac{1}{2}$ for $0 \le x \le 2$, and 0 otherwise.

(c) $E[(Y - g^*(X))^2]$ and $E[\mathrm{Var}(Y|X)]$ are the same thing.

$$\begin{aligned} E[(Y - g^*(X))^2] &= \int_x \int_y (y - E[Y|X = x])^2 f_{Y|X}(y|x) f_X(x) dy dx \\ &= \int_x \mathrm{Var}(Y|X) f_X(x) dx \\ &= E[\mathrm{Var}(Y|X)] \end{aligned}$$

For any given value of X, $f_{Y|X}(y|x)$ is uniform. When $0 \le x \le 1$, $f_{Y|X}(y|x)$ is uniform over $0 \le y \le 1$, and zero otherwise. When $1 \le x \le 2$, $f_{Y|X}(y|x)$ is uniform over $x - 1 \le y \le x$, and zero otherwise. Thus,

$$\mathrm{Var}(Y|X = x) = \begin{cases} \frac{(1-0)^2}{12} = \frac{1}{12} & 0 \le x < 1 \\ \frac{(x-(x-1))^2}{12} = \frac{1}{12} & 1 \le x \le 2 \\ Undefined & Otherwise \end{cases}$$

$$\begin{aligned} E[\mathrm{Var}(Y|X)] &= P(0 \le x < 1)\frac{1}{12} + P(1 \le x \le 2)\frac{1}{12} \\ &= [P(0 \le x < 1) + P(1 \le x \le 2)]\frac{1}{12} \\ &= 1 * \frac{1}{12} \end{aligned}$$

Therefore, $E[(Y - g^*(X))^2] = E[\mathrm{Var}(Y|X)] = \frac{1}{12}$

(d) By the law of conditional variance, we have $\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X])$. Using the answers to (b) and (c),

$$\begin{aligned} \mathrm{Var}(Y) &= E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X]) \\ &= \frac{1}{12} + \frac{5}{48} = \frac{3}{16} \end{aligned}$$

Of course, you can always find $f_Y(y)$ first and then calculate the variance in the usual way; it's just that in this problem we happen to have both components in the sum above.

3. (a) No. Since X_i for any $i \ge 1$ is uniformly distributed between -1.0 and 1.0.

(b) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned} \lim_{i\to\infty} \mathbf{P}(|Y_i - 0| > \epsilon) &= \lim_{i\to\infty} \mathbf{P}(|\frac{X_i}{i} - 0| > \epsilon) \\ &= \lim_{i\to\infty} [\mathbf{P}(X_i > i\epsilon) + \mathbf{P}(X_i < -i\epsilon)] = 0. \end{aligned}$$

(c) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned} \lim_{i\to\infty} \mathbf{P}(|Z_i - 0| > \epsilon) &= \lim_{i\to\infty} \mathbf{P}(|(X_i)^i - 0| > \epsilon) \\ &= \lim_{i\to\infty} [\mathbf{P}(X_i > \epsilon^{\frac{1}{i}}) + \mathbf{P}(X_i < -(\epsilon)^{\frac{1}{i}})] \\ &= \lim_{i\to\infty} [\frac{1}{2}(1 - \epsilon^{\frac{1}{i}}) + \frac{1}{2}(1 - \epsilon^{\frac{1}{i}})] \\ &= 0. \end{aligned}$$

(d) No. In order for T_i to converge in probability, $T_i - T_{i-1}$ must converge to zero in probability. Since $T_i - T_{i-1} = X_i$ for all i, $T_i - T_{i-1}$ does not converge to zero, and therefore T_i does not converge in probability.

(e) Yes, to 0. Applying weak law of large numbers, we have

$$\mathbf{P}(|U_i - \mu| > \epsilon) \to 0 \text{ as } i \to \infty, \text{ for all } \epsilon > 0$$

Here $\mu = 0$ since $X_i \sim U(-1.0, 1.0)$.

(f) Yes, to 0.

$$\begin{aligned}
\mathbf{E}[V_n] &= \mathbf{E}[\mathbf{E}[V_n|X_n]] \\
&= \mathbf{E}[X_n\mathbf{E}[V_{n-1}]] = \mathbf{E}[X_n]\mathbf{E}[V_{n-1}] = 0 \\
\mathrm{var}(V_n) &= \mathbf{E}[\mathrm{var}(V_n|X_n)] + \mathrm{var}(\mathbf{E}[V_n|X_n]) \\
&= \mathbf{E}[X_n^2\mathrm{var}(V_{n-1})] + \mathrm{var}(X_n\mathbf{E}[V_{n-1}]) \\
&= \mathbf{E}[X_n^2]\mathrm{var}(V_{n-1}) + \mathbf{E}[V_{n-1}]^2\mathrm{var}(X_n) \\
&= \frac{1}{3}\mathrm{var}(V_{n-1}) = \left(\frac{1}{3}\right)^{n-1}\mathrm{var}(X_1)
\end{aligned}$$

Notice that as n becomes very large, $\mathrm{var}(V_n)$ approaches 0. By Chebyshev's inequality, we know V_n approaches $\mathbf{E}[V_n] = 0$ in probability.

(g) Yes, to 1. Since for $\epsilon > 0$,

$$\begin{aligned}
\lim_{i\to\infty}\mathbf{P}(|W_i - 1| > \epsilon) &\leq \lim_{i\to\infty}\mathbf{P}(|\max\{X_1,\cdots,X_i\} - 1| > \epsilon) \\
&= \lim_{i\to\infty}[\mathbf{P}(\max\{X_1,\cdots,X_i\} > 1+\epsilon) \\
&\quad +\mathbf{P}(\max\{X_1,\cdots,X_i\} < 1-\epsilon)] \\
&= \lim_{i\to\infty}[0 + (1-\frac{\epsilon}{2})^i] \\
&= 0.
\end{aligned}$$