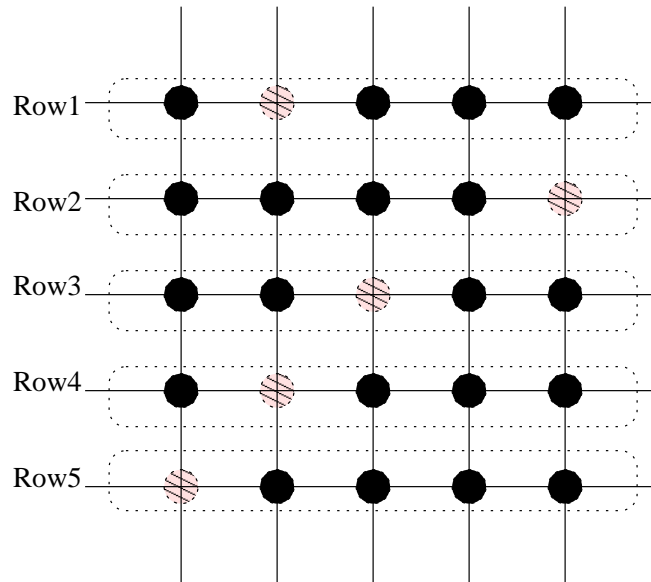


Tutorial 1: Solutions
February 16-17, 2006

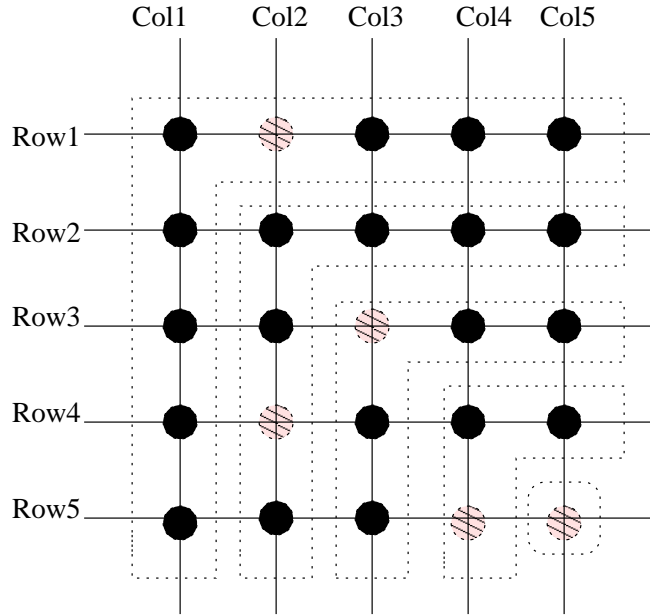
1. Twenty-five pebbles are arranged in five rows of five pebbles each.



- (a) Find the probability that the five pebbles chosen are from different rows:

There are total of $(25)(24)(23)(22)(21) = \frac{25!}{20!}$ ways to get 5 pebbles from 25 pebbles, since there are 25 ways to choose the first pebble and 24 ways to choose the second pebble, etc. If we want to choose five pebbles such that they are in different rows, the number of ways to pick out the first pebble is 25 using counting principle. Since the second pebble can not be on the same row as the first pebble, the number of ways to pick out the second pebble is $25-5=20$. Similarly, the number of ways to choose the third pebble is $20-5=15$, because it can not be on the same row as the first pebble and the second pebble. Using the similar logic, the number of ways to pick the fourth pebble is 10, and finally the number of ways to pick the fifth pebble is 5. Therefore, the probability of choosing 5 pebbles such that they lie on different rows is $\frac{(25)(20)(15)(10)(5)}{\frac{25!}{20!}}$.

- (b) Find the probability that the five pebbles chosen are from different rows and from different columns:



Looking at the 5x5 grids, there are five places to put the first pebble in the first row. After the first pebble is placed, there are only 4 places left to place the second pebble on the second row, since the column used by the first pebble cannot be used by the second pebble. Similarly, there are 3 places to put the third pebble in the third row and 2 places to place the fourth pebble on the fourth row. Finally, there is only one place left to place the last pebble in the last row. So, the total number of possible placements for a particular row arrangement is $(5)(4)(3)(2)(1)$. Notice there are $5!$ ways to arrange 5 rows. So the total number of ways to pick 5 pebbles so that they are in different rows and different columns is $(5!)(5!)$. The number of ways to pick 5 pebbles from 25 pebbles is $\frac{25!}{20!}$. So the probability is $\frac{5!5!}{\frac{25!}{20!}}$.

An alternative way to think of this problem is the following. There are 25 places you can pick the first pebble. Blocking the entire row and the entire column of the first pebble leaves us 16 places to place the second pebble. We take the row and the column out because none of the pebbles can be placed on the same row or the same column. Again blocking the row and column of the second pebble leaves us 9 places to place the third pebble. For the fourth pebble, there are 4 places left, and 1 place for the last pebble. Thus, the total number of ways to pick 5 pebbles with different rows and columns is $(25)(16)(9)(4)(1)$. The probability is $\frac{(25)(16)(9)(4)(1)}{\frac{25!}{20!}}$.

2. (a) The sum of the PMF of a random variable over the possible values that it can take must be equal to 1. Hence, we have

$$1 = \sum_{x=-3}^3 p_X(x) = \frac{9}{a} + \frac{4}{a} + \frac{1}{a} + \frac{1}{a} + \frac{4}{a} + \frac{9}{a} = \frac{28}{a},$$

which implies that $a = 28$. The expected value of random variable is given by

$$\mathbf{E}[X] = \sum_x x p_X(x) = \sum_{x=-3}^3 x \cdot \frac{x^2}{a} = -3 \cdot \frac{9}{a} - 2 \cdot \frac{4}{a} - 1 \cdot \frac{1}{a} + 1 \cdot \frac{1}{a} + 2 \cdot \frac{4}{a} + 3 \cdot \frac{9}{a} = 0.$$

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(In fact, the expected value of a random variable is always equal to 0 if its PMF is even [i.e. $p_X(x) = p_X(-x)$ for all x], and it is a simple exercise to confirm this.)

- (b) The following table shows the value of Z for a given value of X and the probability of that event.

x	-3	-2	-1	0	1	2	3
$p_X(x)$	9/28	1/7	1/28	0	1/28	1/7	9/28
$Z X=x$	9	4	1	0	1	4	9

We see that Z can take only three possible values with non-zero probability, namely 1, 4, and 9. In addition, for each value, there corresponds two values of X . So we have, for example, $p_Z(9) = \mathbf{P}(Z = 9) = \mathbf{P}(X = -3) + \mathbf{P}(X = 3) = p_X(-3) + p_X(3)$. Hence the PMF of Z is given by

$$p_Z(z) = \begin{cases} 1/14 & \text{if } z = 1, \\ 2/7 & \text{if } z = 4, \\ 9/14 & \text{if } z = 9, \\ 0 & \text{otherwise.} \end{cases}$$

- (c) Recall that $\text{var}(X) = \mathbf{E}[(X - \mathbf{E}[X])]^2$, so $\text{var}(X) = \mathbf{E}[Z] = 1 \cdot \frac{1}{14} + 4 \cdot \frac{2}{7} + 9 \cdot \frac{9}{14} = 7$.

3. (a) Using basic properties of the expectation and variance, and the independence of X_1 , X_2 , and X_3 ,

$$\begin{aligned} E[U] &= E[aX_1 + bX_2 + c] = aE[X_1] + bE[X_2] + c \\ E[V] &= E[a'X_1 + b'X_3 + c'] = a'E[X_1] + b'E[X_3] + c' \\ \text{var}(U) &= \text{var}(aX_1 + bX_2 + c) = a^2\text{var}(X_1) + b^2\text{var}(X_2) \\ \text{var}(V) &= \text{var}(a'X_1 + b'X_3 + c') = a'^2\text{var}(X_1) + b'^2\text{var}(X_3) \end{aligned}$$

- (b) First, find a simpler formula for this in terms of U and V :

$$\begin{aligned} E[(U - E[U])(V - E[V])] &= E[UV - VE[U] - UE[V] + E[U]E[V]] \\ &= E[UV] - 2E[U]E[V] + E[U]E[V] \\ &= E[UV] - E[U]E[V] \end{aligned}$$

Now, plugging in what we found above,

$$\begin{aligned} &= E[(aX_1 + bX_2 + c)(a'X_1 + b'X_3 + c')] - (aE[X_1] + bE[X_2] + c)(a'E[X_1] + b'E[X_3] + c') \\ &= E[aa'X_1^2 + ab'X_1X_3 + ac'X_1 + a'bX_1X_2 + bb'X_2X_3 + bc'X_2 + a'cX_1 + b'cX_3 + cc'] - \\ &\quad [aa'(E[X_1])^2 + ab'E[X_1]E[X_3] + ac'E[X_1] + a'bE[X_1]E[X_2] + bb'E[X_2]E[X_3] + bc'E[X_2] + \\ &\quad a'cE[X_1] + b'cE[X_3] + cc'] \end{aligned}$$

Collecting terms,

$$\begin{aligned} &= aa'(E[X_1^2] - (E[X_1])^2) + ab'(E[X_1X_3] - E[X_1]E[X_3]) + a'b(E[X_1X_2] - E[X_1]E[X_2]) + \\ &\quad bb'(E[X_2X_3] - E[X_2]E[X_3]) \end{aligned}$$

Because X_1, X_2, X_3 are independent, only the first term is nonzero:

$$\begin{aligned} &= aa'[E[X_1^2] - (E[X_1])^2] \\ &= aa'var(X_1) \end{aligned}$$

Similarly,

$$E[(U - E[U])(W - E[W])] = a''bvar(X_2)$$

(c) $E[(V - E[V])(W - E[W])] = bb'var(X_3)$

Since b, b' , and $Var(X_3)$ are all strictly positive, this will never be 0, and V and W are necessarily correlated.

(d) Let X and Y be two independent random variables.

$$\begin{aligned} E[(X - E[X])(Y - E[Y])] &= E[(X - E[X])E[(Y - E[Y])] \\ &= (E[X] - E[X])(E[Y] - E[Y]) \\ &= 0 \end{aligned}$$

Therefore, X and Y are uncorrelated.