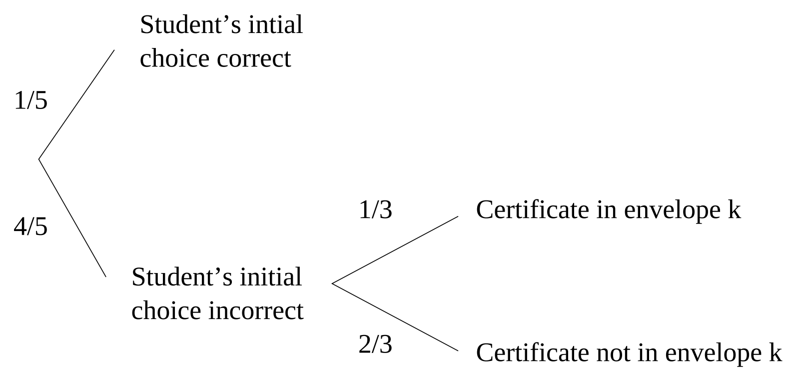
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Tutorial 1: Solutions
February 16-17, 2006

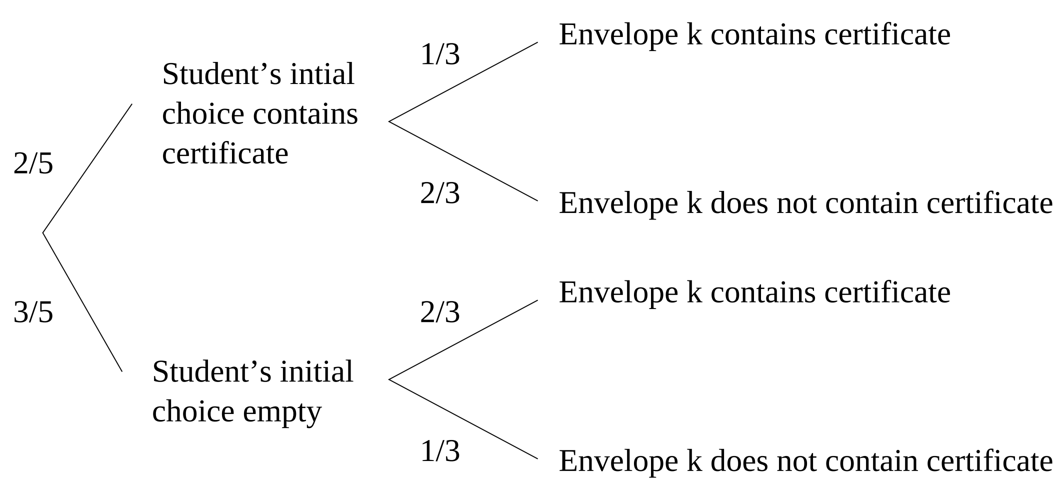
1. (a) The student's initial choice of an envelope is completely random, because the student has no information about which envelope contains the certificate. The probability that the certificate is in the envelope which the student initially picked is therefore $\frac{1}{5}$. The professor now opens one of the other envelopes, but the probability that the student's initial choice was correct remains $\frac{1}{5}$.

   (b) We use the following tree representation:



   If the student's initial choice was correct, the certificate is not in envelope k. If the student's initial choice was incorrect, then the certificate must be in one of the three envelopes which the student has not chosen and the professor has not opened. It has equal probability to be in any of these. The probability that it is in envelope k is therefore $(\frac{4}{5})(\frac{1}{3}) = \frac{4}{15}$.

   (c) When there are two certificates among the five envelopes, the probability that the student selects an envelope with a certificate on her first try is $\frac{2}{5}$. Our tree becomes:



   If the student chose a certificate with an envelope initially, then one of the three envelopes which the student has not chosen and the professor has not opened contains a certificate. If the student did not choose a certificate with an envelope initially, then two of the three envelopes which the student has not chosen and the professor has not opened contain a certificate. The total probability that the certificate is in envelope k is $(\frac{2}{5})(\frac{1}{3})+(\frac{3}{5})(\frac{2}{3}) = \frac{8}{15}$.

(d) One could draw a tree as we have done above, but in this case that's not necessary. The student will find a certificate by the end of the game if and only if one of the two envelopes she chose contained the certificate. The student might as well just point to two envelopes at the same time. The probability that the certificate lies behind one of these two envelopes is $\frac{2}{5}$.

(e) Now there are gift certificates behind two envelopes, and the student chooses two envelopes. There are a total of $\binom{5}{2} = 10$ ways in which a student can pick two envelopes (order does not matter). They are all equally likely. Because there are two gift certificates, there are three empty envelopes. There are $\binom{3}{2} = 3$ ways in which the student can pick two of these. The probability that the student's two envelopes will both be empty is therefore $\frac{3}{10}$. The probability that there will be a gift certificate behind at least one of them (and that the student will win) is $1 - \frac{3}{10} = \frac{7}{10}$.

2. Let M be the event that the person in question is truly a member of the society; then M^c is the event that they are an imposter. Let R be the event that the person in question is recognized as a member.

   In terms of these events, the question gives us the following information (and implications):

$$\begin{aligned} P(M) = 0.01 \quad &\Rightarrow \quad P(M^c) = 0.99 \\ P(R^c|M) = 0.12 \quad &\Rightarrow \quad P(R|M) = 0.88 \\ P(R^c|M^c) = 0.93 \quad &\Rightarrow \quad P(R|M^c) = 0.07 \end{aligned}$$

   Our goal is to determine $P(M|R)$, which we may find by means of Bayes' Rule:

$$\begin{aligned} P(M|R) &= \frac{P(M \cap R)}{P(R)} \\ &= \frac{P(M)P(R|M)}{P(M)P(R|M) + P(M^c)P(R|M^c)} \\ &= \frac{(0.01)(0.88)}{(0.01)(0.88) + (0.99)(0.07)} \\ &\approx \boxed{0.1127} \end{aligned}$$

3. The probability that persons 1 and 2 both roll a particular face is $1/n^2$. Therefore,

$$P(A_{12}) = P(A_{13}) = P(A_{23}) = \frac{n}{n^2} = \frac{1}{n}.$$

   Similarly, we also have

$$P(A_{12} \cap A_{13}) = P(\text{all players roll the same face}) = \frac{n}{n^3} = \frac{1}{n^2},$$

   so

$$P(A_{12} \cap A_{13}) = P(A_{12}) \cdot P(A_{13})$$

   . Hence A_{12} and A_{13} are independent, and the same is true of any other pair from the events A_{12}, A_{13}, and A_{23}. However, A_{12}, A_{13}, and A_{23} are not independent. In particular, if A_{12} and A_{13} occur, then A_{23} also occurs.