Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
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Tutorial 07 Solutions
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1. The problem is simplified by looking at the fraction of the original stake that the gambler has at any given moment. Because the expected value operation is linear, we can compute the expected fraction of the original stake and multiply by the original stake to get the expected total fortune (the original stake is a constant).

   If the gambler has a at the beginning of a round, he bets $a(2p-1)$ on the round. If he wins, he'll have $a + a(2p-1)$ units. If he loses, he'll have $a - a(2p-1)$ units. Thus at the end of the round, he will have $2pa$ following a win, and $2(1-p)a$ following a loss.

   Thus, we see that winning multiplies the gambler's fortune by $2p$ and losing multiplies it by $2(1-p)$. Therefore, if he wins k times and loses m times, he will have $(2p)^k(2(1-p))^m$ times his original fortune. We can also compute the probability of this event. Let Y be the number of times the gambler wins in the first n gambles. Then Y has the binomial PMF:

   $$p_Y(y) = \binom{n}{y} p^y (1-p)^{n-y}, \qquad y = 0, 1, \ldots, n.$$

   We can now calculate the expected fraction of the original stake that he has after n gambles. Let Z be a random variable representing this fraction. We know that Z is related to Y via

   $$Z = (2p)^Y (2(1-p))^{n-Y}.$$

   We will calculate the expected value of Z using the PMF of Y.

   $$\begin{aligned}
   E[Z] &= \sum_{y=0}^{n} Z(y) p_Y(y) = \sum_{y=0}^{n} (2p)^y [2(1-p)]^{n-y} \binom{n}{y} p^y (1-p)^{n-y} \\
   &= \sum_{y=0}^{n} 2^y p^y 2^{n-y} (1-p)^{n-y} \binom{n}{y} p^y (1-p)^{n-y} \\
   &= 2^n \sum_{y=0}^{n} p^y (1-p)^{n-y} \binom{n}{y} p^y (1-p)^{n-y} \\
   &= 2^n \sum_{y=0}^{n} \binom{n}{y} \left(p^2\right)^y \left[(1-p)^2\right]^{n-y} \\
   &= 2^n \left(p^2 + (1-p)^2\right)^n,
   \end{aligned}$$

   where the last equality follows using the generalized binomial formula

   $$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n.$$

   Thus the gambler's expected fortune is

   $$2^n \left(p^2 + (1-p)^2\right)^n x,$$

   where x is the fortune at the beginning of the first round.

An alternative method for solving the problem involves using iterated expectations. Let X_k be the fortune after the kth gamble. Again, we use the fact that the expected fortune after the kth gamble is

$$X_k = 2\left(p^2 + (1-p)^2\right) X_{k-1}.$$

Therefore, using iterated expectations, the fortune after n gambles is

$$\begin{aligned}
E[X_n] &= E\left[E[X_n|X_{n-1}]\right] \\
&= 2\left(p^2 + (1-p)^2\right) E[X_{n-1}] \\
&= 2\left(p^2 + (1-p)^2\right) E\left[E[X_{n-1}|X_{n-2}]\right] \\
&= \left(2\left(p^2 + (1-p)^2\right)\right)^2 E[X_{n-2}] \\
&= \left(2\left(p^2 + (1-p)^2\right)\right)^2 E\left[E[X_{n-2}|X_{n-3}]\right] \\
&= \left(2\left(p^2 + (1-p)^2\right)\right)^3 E[X_{n-3}] \\
&= \cdots \\
&= \left(2\left(p^2 + (1-p)^2\right)\right)^n E[X] \\
&= 2^n\left(p^2 + (1-p)^2\right)^n x.
\end{aligned}$$

2. a) X, Y cannot be independent, since given X we know the value of Y to within two values, and hence it is easy to show that:

$$f(x|y) \neq f(x).$$

b) Y, Z are independent because X is symmetric about around the ordinate (i.e. what we typically call the Y-axis).

c)

$$\begin{aligned}
f_{YZ}(y,z) &= f_{Y|Z}(y|z) \cdot f_Z(z) \\
&= f_X(x) \cdot f_Z(z)
\end{aligned}$$

and therefore:

$$f_Y(y) = \sum_I f_X(x) \cdot f_Z(z) = f_X(x)$$

and therefore $Y \sim N[0,1]$ as desired.

d) We want to show that $cov(X,Y) = 0$. Since $E[X] = E[Y] = 0$, we have:

$$\begin{aligned}
cov(X,Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{Y|X}(y|x) \cdot f_X(x)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)\frac{1}{2}(\delta(x) + \delta(-x) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} y[xf_X(x) - xf_X(x)] \\
&= 0
\end{aligned}$$

as required. The last equality follows from the fact that since X is a standard normal random variable, $f_X(x) = f_X(-x)$. Note that we have two dependent normal random variables X, Y that have zero correlation. There is a small subtlety here. We know that if two random variables have bivariate joint distribution, and are uncorrelated, then they are independent. However in this case, we have two dependent normal random variables, whose correlation is zero. The difference here is that the joint distribution is not bivariate normal.

3. (a) Let K be the random variable for the number of gaurds that you bumped into on your way. K has a binomial distribution with parameters $n = 78$ and $p = 1/2$. Hence,

$$p_K(k) = \begin{cases} \binom{78}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{78-k} & k = 0, 1, \ldots, 78 \\ 0 & otherwise \end{cases}$$

Given that you bumped into k guards, random variable X is the sum of k independent normal random variables, each with mean 1 and standard deviation of 1/2. Therefore, conditioned on $K = k$, X is is a normal random variable with mean k and standard deviation $(1/2)\sqrt{k}$:

$$f_{X|K}(x|k) = \frac{1}{\sqrt{2\pi}\frac{1}{2}\sqrt{k}} e^{-\frac{(x-k)^2}{\frac{1}{2}k}} \quad (1)$$

$$= \sqrt{\frac{2}{\pi k}} e^{-\frac{2(x-k)^2}{k}} \quad (2)$$

The transform of X conditioned on $K = k$ is therefore

$$E[e^{sX}|K = k] = e^{(ks^2/8)+ks} \quad (3)$$

Using the total probability theorem, we get:

$$f_X(x) = \sum_{k=0}^{78} p_K(k) f_{X|K}(x|k) \quad (4)$$

$$= \sum_{k=0}^{78} \binom{78}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{78-k} \sqrt{\frac{2}{\pi k}} e^{-\frac{2(x-k)^2}{k}} \quad (5)$$

Similarly, since we know the conditional transform of X given $K = k$, we can use the total expectation theorem to get

$$M_X(s) = \sum_{k=0}^{78} p_K(k) E[e^{sX}|K = k] \quad (6)$$

$$= \sum_{k=0}^{78} \binom{78}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{78-k} e^{(ks^2/8)+ks} \quad (7)$$

Thus X is a mixture of normal random variables, and its transform is a mixture of the corresponding normal transforms. Note, however, that X itself is not normal!

(b) Let K be the number of gaurds that you bump into. We can view X as the sum of K independent normal random variables, each with mean 1 and standard deviation of 1/2. Thus the transform associated with X can be found by replacing in the binomial transform $M_K(s) = (\frac{1}{2} + \frac{1}{2}e^s)^{78}$ the occurrences of e^s by the normal transform corresponding to $\mu = 1$ and $\sigma = \frac{1}{2}$. Thus,

$$M_X(s) = \left(\frac{1}{2} + \frac{1}{2}\left(e^{\frac{s^2}{8}+s}\right)\right)^{78}.$$