6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013
Transcript – Tutorial: The Absent Minded Professor

Hi. In this problem, we have an absent-minded professor who will inadvertently give us some practice with exponential random variables. So the professor has made two appointments with two students and inadvertently made them at the same time. And what we do is we model the duration of these appointments with an exponential random variable.

So remember, an exponential random variable is a continuous random variable that takes on non-negative values, and it's parametrized by a rate parameter, lambda. And the exponential random variable is often used to model durations of time-- so time until something happens, so for example, in this case, time until the student leaves or the appointment is over. Or sometimes you will also use it to be as a model of time until something fails.

And one thing that will be useful is the CDF of this exponential random variable. So the probability that it's less than or equal to some value, little t, is equal to 1 minus e to the minus lambda t. So this is, of course, valid only when t is non-negative.

The other useful property is that the expected value of an exponential random variable is just 1 over the parameter lambda. And the last thing that we'll use specifically in this problem is the memory list property of exponential random variables. And so recall that that just means that if you pop in the middle of an exponential random variable, the distribution for that random variable going forward from the point where you popped in is exactly the same as if it had just started over. So that's why we call it the memory list property. Basically, the past doesn't really matter, and going forward from whatever point that you observe it at, it looks as if it had just started over afresh.

And last thing we'll use, which is a review of a concept from earlier, is total expectation. So let's actually model this problem with two random variables. Let's let T1 be the time that the first student takes in the appointment and T2 be the time that the second student takes. And what we're told in the problem is that they're both exponential with mean 30 minutes.

So remember the mean being 30 minutes means that the lambda is 1 over the mean. And so the lambda in this case would be 1/30. And importantly, we're also told that they are independent. So how long the first person takes is independent of how long the second person takes.

So the first student arrives on time and take some random amount of time, T1. The second student arrives exactly five minutes late. And whatever the second person meets with the professor, that student will then take some random amount of time, T2. What we're asked to do is find the expected time between when the first student arrives-- so we can just call that time 0-- and when the second student leaves.

Now you may say, well we're dealing with expectations, so it's easier. And in this case, it probably is just the expectation of how long the first student takes plus the expectation of how long the second student takes. So it should be about 60 minutes or exactly 60 minutes. Now, why is that not exactly right? It's because there is a small wrinkle, that the students may not go exactly back to back.

So let's actually draw out a time frame of what might actually happen. So here's time 0, when the first student arrives. And the first will go for some amount of time and leave.

And now let's consider two scenarios. One scenario is that the first student takes more than five minutes to complete. Well then the second student will have arrived at 5 minutes and then will be already waiting whenever this first student leaves. So then the second student will immediately pick up and continue. And in that case, we do have two exponentials back to back.

But there could be another situation. Suppose that the first student didn't take very long at all and finished within five minutes, in which case the second student hasn't arrived yet. So this professor is idle in between here. And so we actually don't necessarily have two of them going back to back. So there's an empty period in between that we have to account for.

So with that in mind, we see that we have two scenarios. And so what does that beg to use? Well, we can split them up into the two scenarios and then calculate expectations with each one and then use total expectation to find the overall expected length of time.

OK, so let's begin with the first scenario. The first scenario is that, let's say, the first student finished within five minutes. So what does that mean in terms of the definitions that we've used? That means T1 is less than or equal to 5.

So if the first student took less than five minutes, then what happens? Then we know that the amount of time that you'd need to take-- let's call that something else. Let's call that X. So X is the random variable that we're interested in, the time between when the first student comes and the second student leaves. This is the value that we want to find.

Well we know that we're guaranteed that there will be a five-minute interval. So first student will come, and then the second person will take over. So we're guaranteed that the first five minutes will be the difference between when time starts and when the second student arrives.

And then, after that, it's just however long the second student takes, which is just the expected value of T2. And T2 is a exponential random variable with mean 30. So in this case, it's just 35.

So the first student doesn't take very long. Then we just get the five minutes, that little buffer, plus however long the second student takes, which, on average, is 30 minutes. Now what is the probability of this happening?

The probability of this happening is the probability that the first student takes less than five minutes. And here is where we use the CDF that we wrote out earlier. It's going to be 1 minus e to the minus lambda t. So in this case, t is five and lambda is 1/30. So it's minus 5/30 is the probability.

All right, now let's consider the second case. The second case is that the first student actually takes longer than five minutes. OK, so what happens in that case?

Here's five minutes. The first student came to five minutes. The second student arrived, and the first student is still going. So he goes for some amount of time. And then whenever he finishes, the second student continues. So now the question is, what is the total amount of time in this case?

Well, you can think of it as using the memory list property. This is where it comes in. So the first five minutes, we know that it was already taken because we're considering the second scenario, which we're given that T1 is greater than 5.

And so the question now is, if we know that, how much longer does it take? How much longer past the five-minute mark does the first student take? And by the member list property, we know that it's as if the first student started over. So there was no memory of the first five minutes, and it's as if the first student just arrived also at the five-minute mark and met with the professor. So past the five-minute mark, it's as if you have a new exponential random variable, still with mean 30.

And so what we get is that, in this case, you get the guaranteed five minutes, and then you get the memory list continuation of the first student's appointment. So you get another 30 minutes on average because of the memory list property. And then whenever the first student finally does finish up, the second student will immediately take over because he has already arrived. It's past the five-minute mark. And then that second student will take, again, on average, 30 more minutes.

So what you get is, in this case, the appointment lasts 65 minutes on average. Now what is the probability of this case? The probability of this case is the probability that T1 is greater than 5. And now we know that that is just the complement of this, 1 minus that. So it's just e to the minus 5/30.

So now we have both scenarios. We have the probabilities of each scenario, and we have the expectation under each scenario. Now all that remains now is to combine them using total expectation.

So I really should have written expectation of X given T1 is less than or equal to 5 here. And this is expectation of X given that T1 is greater than 5. So expectation of X overall is the probability that T1 is less than or equal to 5 times the expectation of X given that T1 is less than or equal to 5 plus the probability that T1 is greater than 5 times the expectation of X given that T1 is greater than 5. And we have all four of these pieces here. so it's 35 times 1 minus e to the minus 5/30 plus 65 times e to the minus 5 5/30. And it turns out that this is approximately equal to 60.394 minutes.

All right, so what have we found? We found that the original guess that we had, if we just had two meetings back to back, was on average it would take 60 minutes. It turns out that, because of the way that things are set up, because of the five minute thing, it actually takes a little longer than 60 minutes on average.

And why is that? It's because the five sometimes adds an extra buffer, adds a little bit of extra amount, because it would have been shorter in this scenario because, if the both students had arrived on time, then the second student would have been able to pick up right here immediately. And so both appointments would have ended sooner. But because the second student didn't arrive until five minutes, there was some empty space that was wasted. And that's where you get you the little bit of extra time.

So this is a nice problem just to get some more exercise with exponential random variables and also nicely illustrates the memory list property, which was a key points in order to solve this. And it also is nice because we get to review a useful tool that we've been using all course long, which is to split things into different scenarios and then solve the simpler problems and then combine them up, for example using total expectation. So I hope that was helpful, and see you next time.