

6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: Mean & Variance of the Exponential

Hi. In this video, we're going to compute some useful quantities for the exponential random variable.

So we're given that x is exponential with rate λ . PDF looks like this, and the formula is here.

First question, part a, what's the CDF? So let's go right in. The CDF of x is the probability that X is less than or equal to little x . Let's look at some cases here.

What if little x is less than 0? Well, x random variable only takes on these non-negative values. And so the probability that X is less than or equal to some negative number is going to be 0.

On the other hand, if x is greater than or equal to 0, we do actually have to integrate here. So to do that, we take the integral from minus infinity to x of f_x of t -- the dummy variable here used is t .

Notice that again, f_x of t is going to be 0 for negative values, so we take the integral here from 0. And now we plug in for f_x of t . That's $-\lambda e^{-\lambda t} dt$. And recall that the integral of u to the a t is $1/a e^{-at}$.

So here in this case, we'll get λ , which is just a constant. And then a here is going to be $-\lambda$. So we get this, 0 to x . λ 's cancel and we actually get $1 - e^{-\lambda x}$. So do this. And we are done with the CDF.

Now for the expectation. We use the standard formula, which is $-\infty$ to ∞ t times f_x of t dt . So again, f_x of t is going to be 0 for a negative value. So we do the integral from 0. We get 0 to ∞ $t \lambda e^{-\lambda t} dt$.

Now, you can try all you want to get rid of this t . It's not going to go even if you try all kinds of u substitution. But at the end the day, you're going to have to pull out your calculus textbook and find the integration by parts formula, which is-- $\int u dv$.

So the hope is that this integral is going to be easier than the one on the left. Notice that this is the integral of one of the terms here. And this is the derivative of one of the terms. So that may help you decide on how you select u and v .

In our case actually, I'm going to use u as-- t for u . Because when you take the derivative, it's going to become 1. And the derivative is what's going to go in that integral.

So this is going to be dt for du . And then, dv I'm going to select as whatever's left over. It's $\lambda e^{-\lambda t} dt$. So v is going to be-- we already did the integral-- $e^{-\lambda t}$ to the minus λt .

And so if we do this, it's going to be negative t times $e^{-\lambda t}$. So that's uv . Minus v , which is $e^{-\lambda t}$ times dt . That goes from 0 to infinity. This is evaluated from 0 to infinity.

Well, what does it mean for this to be evaluated from 0 to infinity? A better and easier way to look at this is to say, well, it's going to go from 0 to x . But then you take the limit as x goes to infinity. So that's going to help us here.

And this negative-- these negatives cancel. And we're left with-- let's plug in the bounds. We're left with negative x minus λx plus the integral of this is going to be 1 over negative $\lambda e^{-\lambda t}$ evaluated from 0 to infinity.

All right, so now the limit. So for the limit, notice that x increases as x goes to infinity. And this exponential decays. So they're kind of competing for each other. But the exponential is going to win because it decays way faster than x . And so this first term is going to go off-- the limit is going to go to 0.

All right. For this, if you evaluate the balance, the infinity makes this 0. And 0, you're going to get 1 over λ . So that's 1 over λ . All right. And so the expectation is 1 over λ .

OK, so now what's the variance? That's part c, right?

So we use the standard formula for variance, which is this. We already figured out the expectation. We just need to figure out the expectation of x squared.

Well, we're just going to follow the same set of steps from before. For x squared, it's just going to be t squared, t squared, t squared, x squared. The only thing that's going to change is what we choose for u here, for the u substitution. So it's going to be t squared. So the derivative is going to change to $2t dt$. v is going to be exactly the same. And so here in this term, we get negative $2t e^{-\lambda t}$. But there's a negative sign out here, so the negatives cancel and we're left with a positive sign here. This is going to change. All right. OK.

So in order to do this integral, we can use a trick. We can move this-- so there's a $2t$ here. We move this 2 in here, leave the t inside. And you have to leave the t inside. But multiply by λ and divide by λ .

Now, look at that integral. 0 to infinity t times $\lambda e^{-\lambda t} dt$. Exactly the expectation that we computed. We already did that. That is just 1 over λ , so it's 2 over λ times 1 over λ .

Again, the limit as x goes to infinity-- the exponential will beat x squared. No matter what polynomial we put in there, the exponential's going to win. So this is going to be 0 still. This

one's going to be 2 over λ squared. So we're left with 2 over λ squared for expectation of x squared. And so we have 1 over λ squared for the variance. OK, so we're done with the variance.

Part d. We're given that x_1 , x_2 , and x_3 are independent and identically distributed. They're exponentials with rate λ . We're asked for the PDF of z , which is the max of x_1 , x_2 , and x_3 . How do we generally find a PDF?

We take the CDF and then take the derivative, right? We first find the CDF, and then take the derivative. So let's do that.

So first, let's see. Part d, find the CDF of z , which is going to be the probability that Z is less than or equal to little z , which is going to be equal to the probability that the max of x_1 , x_2 , x_3 is less than or equal to z . And this is going to have the same sort of structure as before.

If z is less than 0 , x_1 , x_2 , x_3 are positive-- non-negative. And so this is the probability that if you get little z less than 0 , you're not going to have any probability there. And so if z is greater than or equal to 0 is where it gets interesting. We need to do something special.

So the special thing here is to recognize that the probability of the max being less than or equal to z is actually also the probability of each of these random variables individually being less than or equal to z . Why is that true?

One way to check whether the events-- these two events are the same is to check the two directions. One direction say, if the max of x_1 , x_2 , x_3 is less than or equal to z , does that mean x_1 is less than or equal to z , x_2 is less than or equal to z , and x_3 is less than or equal to z ?

Yes. OK.

And then, if x_1 , x_2 , and x_3 are individually less than or equal to z , then the max is also less than or equal to z . So these two events are equivalent and this is true. By independence we can break this up. And we get-- these are all CDFs of the exponential and they all have this form. So it's just going to be 1 minus e to the minus λz cubed. Plug this in here. And then, try to take the derivative to get the PDF.

Let's see. So it's going to be the same, like this for z less than 0 . For z greater than or equal to 0 , it's going to be the derivative of this thing. Derivative of this thing is by chain rule, 3 times 1 minus e to the minus λz squared. Then the derivative of negative e to the minus λz , that's just λe to the minus λz . There we go. This is the PDF we were looking for.

So last problem. We're looking for the PDF of w , which is the min of x_1 and x_2 . So let's try this as a similar approach. Try the same thing, actually. See if it works.

So w , w , w , w , min, less than or equal to w . OK. So let's see if this works.

Is it true that the min-- if the min of x_1 and x_2 is less than or equal to w , that each of them is less than or equal to w ? No, right?

X_1 could be less than or equal to w and x_2 could be bigger than w . And the min could still be less than or equal to w . So that's definitely not true. So what do we do here?

The trick is to flip it and say we want to compute the min of x_1 and x_2 being greater than w . In that case, let's check if we can do this trick.

If the min of x_1 and x_2 is greater than w , then clearly x_1 is bigger than w and x_2 is bigger than w . And if x_1 and x_2 are individually bigger than w , then clearly the min's also bigger than w . So this works. And now we can use independence as before.

And for this, this is just 1 minus the CDF here. So it's just going to be e to the minus λw for each of them. But that's the same as e to the minus $2\lambda w$. Or e to the $-2\lambda w$. So it's going to be--

Notice the similarity between this and this. The only difference is this has a 2λ in there. That means that w is an exponential random variable with rate 2λ .

So then the PDF is going to be an exponential, whatever it is for an exponential. Except with rate 2λ .

You can also take the derivative of this and find that you get this. OK, so we're done with the problems. We computed some interesting quantities for the exponential random variable in this--

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