6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013
Transcript – Recitation: Uniform Probabilities on a Triangle

Hi. In this problem, we're going to get a bunch of practice working with multiple random variables together. And so we'll look at joint PDFs, marginal PDFs, conditional PDFs, and also get some practice calculating expectations as well. So the problem gives us a pair of random variables-- x and y. And we're told that the joint distribution is uniformly distributed on this triangle here, with the vertices being 0, 0 1, 0, and 0, 1. So it's uniform in this triangle.

And the first part of the problem is just to figure out what exactly is disjoint PDF of the two random variables. So in this case, it's pretty easy to calculate, because we have a uniform distribution. And remember, when you have a uniform distribution, you can just imagine it being a sort of plateau coming out of the board. And it's flat.

And so the height of the plateau, in order to calculate it, you just need to figure out what the area of this thing is, of this triangle is. So remember, when you had single random variables, what we had to do was calculate, for uniform distribution, we had to integrate to 1. So you took the length, and you took 1 over the length was the correct scaling factor.

Here, you take the area. And the height has to make it so that the entire volume here integrates to 1. So the joint PDF is just going to be 1 over whatever this area is. And the area is pretty simple to calculate. It's 1/2 base times height. So it's 1/2.

And so what we have is that the area is 1/2. And so the joint PDF of x and y is going to equal 2. But remember, you always have to be careful when writing these things to remember the ranges when these things are valid. So it's only 2 within this triangle.

And outside of the triangle, it's 0. So what exactly does inside the triangle mean? Well, we can write it more mathematically. So this diagonal line, it's given by x plus y equals 1. So everything in the triangle is really x plus y is less than or equal to 1. It means everything under this triangle.

And so we need x plus y to be less then or equal to 1 and also x to be non-negative and y to be non-negative. So with these inequalities, that captures everything within this triangle. And otherwise, the joint PDF is going to be 0.

The next part asks us to find, using this joint PDF, the marginal of y. And remember, when you have a joint PDF of two random variables, you essentially have everything that you need, because from this joint PDF, you can calculate marginals, you can calculate from the margins, you can calculate conditionals. The joint PDF captures everything that there is to know about this pair of random variables.

Now, to calculate a marginal PDF of y, remember a marginal really just means collapsing the other random variable down. And so you can just imagine taking this thing and collapsing it down onto the y-axis. And mathematically, that is just saying that we integrate out the other random variable.

So the other random variable in this case will be x. We take x and we get rid of it by integrating out from negative infinity to infinity. Of course, this joint PDF is 0 in a lot of places. And so a lot of these will be 0. And only for a certain range of x's will this integral actually be non-zero.

And so again, the other time when we have to be careful is when we have these limits of integration, we need to make sure that we have the right limits. And so we know that the joint PDF is 2. It's nonzero only within this triangle. And so it's only 2 within this triangle, which means what for x?

Well, depending on what x and y are, this will be either 2 or 0. So let's just fix some value of y. Pretend that we've picked some value y, let's say here. We want this value of y.

Well, what are the values of x such that the joint PDF for that value y is actually nonzero, it's actually 2? Well, it's everything from x equals 0 to whatever x value this is. But this x value, actually, if you think about it, is just 1 minus y, because this line is x plus y equals 1. So whatever y is, x is going to be 1 minus that.

And so the correct limits would actually be from 0 to 1 minus y. And then the rest of that is pretty simple. You integrate this. This is a pretty simple integral. And you get that it's actually two times 1 minus y. That's a y.

But of course, again, we need to make sure that we have the right regions. So this is not always true for y, of course. This is only true for y between 0 and 1. And otherwise, it's actually 0, because when you take a y down here, well, there's no values of x that will give you a nonzero joint PDF.

And if you take a value of y higher than this, the same thing happens. So we can actually draw this out and see what it looks like. So let's actually draw a small picture here. Here's y.

Here's the marginal PDF of y. And here's 2. And it actually looks like this. It's a triangle and a 0 outside this range.

So does that make sense? Well, first of all, you see that actually does in fact integrates to 1, which is good. And the other thing we notice is that there is a higher density for smaller values of y. So why is that?

Why are smaller values of y more likely than larger values of y? Well, because when you have smaller values of y, you're down here. And it's more likely because there are more values of x that go along with it that make that value of y more likely to appear.

Say you have a large value of y. Then you're up here at the tip. Well, there aren't very many combinations of x and y that give you that large a value of y. And so that large value of y becomes less likely.

Another way to think about it is, when you collapse this down, there's a lot more stuff to collapse down its base. There's a lot of x's to collapse down. But up here, there's only a very little bit of x to collapse down. And the PDF of y becomes more skewed towards smaller values of y.

So now, the next thing that we want to do is calculate the conditional PDF of x, given y. Well, let's just recall what that means. This is what we're looking for-- the conditional PDF of x, given y.

And remember, this is calculated by taking the joint and dividing by the marginal of y. So we actually have the top and the bottom. We have to joint PDF from part A. And from part B, we calculated the marginal PDF of y. So we have both pieces.

So let's actually plug them in. Again, the thing that you have to be careful here is about the ranges of x and y where these things are valid, because this is only non-zero when x and y fall within this triangle. And this is only non-zero when y is between 0 and 1. So we need to be careful.

So the top, when it's non-zero, it's 2. And the bottom, when it's non-zero, it's 2 times 1 minus y. So we can simplify that to be 1 over 1 minus y. And when is this true?

Well, it's true when x and y are in the triangle and y is between 0 and 1. So put another way, that means that this is valid when y is between 0 and 1 and x is between 0 and 1 minus y, because whatever x has to be, it has to be such that they actually still fall within this triangle. And outside of this, it's 0.

So let's see what this actually looks like. So this is x, and this is the conditional PDF of x, given y. Let's say this is 1 right here.

Then what it's saying is, let's say we're given that y is some little y. Let's say it's somewhere here. Then it's saying that the conditional PDF of x given y is this thing. But notice that this value, 1 over 1 minus y, does not depend on x. So in fact, it actually is uniform.

So it's uniform between 0 and 1 minus y. And the height is something like 1 over 1 minus y. And this is so that the scaling makes it so that actually is a valid PDF, because the integral is to 1.

So why is the case? Why is that when you condition on y being some value, you get that the PDF of x is actually uniform? Well, when you look over here, let's again just pretend that you're taking this value of y.

Well, when you're conditioning on y being this value, you're basically taking a slice of this joint PDF at this point. But remember, the original joint PDF was uniform. So when you take a slice of a uniform distribution, joint uniform distribution, you still get something that is uniform. Just imagine that you have a cake that is flat.

Now, you take a slice at this level. Then whatever slice you have is also going to be imagine being a flat rectangle. So it's still going to be uniform. And that's why the conditional PDF of x given y is also uniform.

Part D now asks us to find a conditional expectation of x. So we want to find the expectation of x, given that y is some little y. And for this, we can use the definition. Remember, expectations are really just weighted sums. Or in the [? continuous ?] case, it's an integral.

So you take the value. And then you weight it by the density. And in this case, because we're taking conditional a expectation, what we weight it by is the conditional density. So it's the conditional density of x given that y is little y. We integrate with respect to x.

And fortunately, we know what this conditional PDF is, because we calculated it earlier in part C. And we know that it's this-- 1 over 1 minus y. But again, we have to be careful, because this formula, 1 over 1 minus y, is only valid certain cases. So let's think about this first.

Let's think about some extreme cases. What if y, little y, is negative? If little y is negative, we're conditioning on something over here. And so there is no density for y being negative or for y, say, in other cases when y is greater than 1.

And so in those cases, this expectation is just undefined, because conditioning on that doesn't really make sense, because there's no density for those values of y. Now, let's consider the case that actually makes, sense where y is between 0 and 1. Now, we're in business, because that is the range where this formula is valid.

So this formula is valid, and we can plug it in. So it's 1 over 1 minus y dx. And then the final thing that we again need to check is what the limits of this integration is. So we're integrating with respect to x. So we need to write down what values of x, what ranges of x is this conditional PDF valid.

Well, luckily, we specified that here. x has to be between 0 and 1 minus y. So let's actually calculate this integral. This 1 over 1 minus y is a constant with respect to x. You can just pull that out.

And then now, you're really just integrating x from 0 to 1 minus y. So the integral of x is [? 1 ?], 1/2x squared. So you get a 1/2x squared, and you integrate that from 0 to 1 minus y. And so when you plug in the limits, you'll get a 1 minus y squared. That will cancel out the 1 over 1 minus y.

And what you're left with is just 1 minus y over 2. And again, we have to specify that this is only true for y between 0 and 1. Now, we can again actually verify that this makes sense. What we're really looking for is the conditional expectation of x given some value of y.

And we already said that condition on y being some value of x is uniformly distributed between 0 and 1 minus y. And so remember for our uniform distribution, the expectation is simple. It's just the midpoint. So the midpoint of 0 and 1 minus y is exactly 1 minus y/2. So that's a nice way of verifying that this answer is actually correct.

Now, the second part of part D asks us to do a little bit more. We have to use the total expectation theorem in order to somehow write the expectation of x in terms of the expectation of y. So the first thing we'll do is use the total expectation theorem.

So the total expectation theorem is just saying, well, we can take these conditional expectations. And now, we can integrate this by the marginal density of y, then we'll get the actual expectation of x. You can think of it as just kind of applying the law of iterated expectations as well. So this integral is going to look like this.

You take the conditional expectation. So this is the expectation of x if y were equal to little y. And now, what is that probability? Well, now we just multiply that by the density of y at that actual value of little y.

And we integrate with respect to y. Now, we've already calculated what this conditional expectation is. It's 1 minus y/2. So let's plug that in. 1 minus y/2 times the marginal of y.

There's a couple ways of attacking this problem now. One way is, we can actually just plug in that marginal of y. We've already calculated that out in part B. And then we can do this integral and calculate out the expectation.

But maybe we don't really want to do so much calculus. So let's do what the problem says and try a different approach. So what the problem suggests is to write this in terms of the expectation of y. And what is the expectation of y?

Well, the expectation of y is going to look something like the integral of y times the marginal of y. So let's see if we can identify something like that and pull it out. Well, yeah, we actually do have that. We have y times the marginal of y, integrated.

So let's isolate that. So besides that, we also have this. We have the integral of the first term, is 1/2 times the marginal of y. And then the second term is minus 1/2 times the integral of y of dy.

This is just me splitting this integral up into two separate integrals. Now, we know what this is. The 1/2 we can pull out. And then the rest of it is just the integral of a marginal of a density from minus infinity to infinity. And by definition, that has to be equal to 1. So this just gives us a 1/2.

And now, what is this? We get a minus 1/2. And now this, we already said that is the expectation of y. So what we have is the expectation of y.

So in the second part of this part D, we've expressed the expectation of x in terms of the expectation of y. Now, maybe that seems like that's not too helpful, because we don't know what either of those two are. But if we think about this problem, and as part E suggests, we can see that there's symmetry in this problem, because x and y are essentially symmetric.

So imagine this is x equals y. There's symmetry in this problem, because if you were to swap the roles of x and y, you would have exactly the same joint PDF. So what that suggests is that by symmetry then, it must be that the expectation of x and the expectation of y are exactly the same.

And that is using the symmetry argument. And that helps us now, because we can plug that in and solve for expectation of x. So expectation of x is 1/2 minus 1/2 expectation of x. So we have 3/2 expectation of x equals 1/2.

So expectation of x equals 1/3. And of course, expectation of y is also 1/3. And so it turns out that the expectation is around there. So this problem had several parts.

And it allowed us to start out from just a raw joint distribution, calculate marginals, calculate conditionals, and then from there, calculate all kinds of conditional expectations and expectations. And a couple of important points to remember are, when you do these joint distributions, it's very important to consider where values are valid.

So you have to keep in mind when you write out these conditional PDFs and joint PDFs and marginal PDFs, what ranges the formulas you calculated are valid for. And that also translates to when you're calculating expectations and such. When you have integrals, you need to be very careful about the limits of your integration, to make sure that they line up with the range where the values are actually valid.

And the last thing, which is kind of unrelated, but it is actually a common tool that's used in a lot of problems is, when you see symmetry in these problems, that can help a lot, because it will simplify things and allow you to use facts like these to help you calculate what the final answer is. Of course, this is also comes along with practice.

You may not immediately see that there could be a symmetry argument that will help with this problem. But with practice, when you do more of these problems, you'll eventually build up that kind of--