6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: The Difference of Two Independent Exponential Random Variables

In this problem, Romeo and Juliet are to meet up for a date, where Romeo arrives at time x and Juliet at time y, where x and y are independent exponential random variables, with parameters lambda. And we're interested in knowing the difference between the two times of arrivals, we'll call it z, written as x minus y. And we'll like to know what the distribution of z is, expressed by the probability density function, f of z.

Now, we'll do so by using the so-called convolution formula that we learn in the lecture. Recall that if we have a random variable w that is the sum of two independent random variables, x plus y, now, if that's the case, we can write the probability [INAUDIBLE] function, fw, [INAUDIBLE] as the following integration-- negative infinity to infinity fx little x times f of y w minus x, integrated over x.

And to use this expression to calculate f of z, we need to do a bit more work. Notice w is expressed as a sum of two random variables, whereas z is expressed as the subtraction of y from x. But that's fairly easy to fix. Now, we can write z. Instead of a subtraction, write it as addition of x plus negative y.

So in the expression of the convolution formula, we'll simply replace y by negative y, as it will show on the next slide. Using the convolution formula, we can write f of z little z as the integration of f of x little x and f of negative y z minus x dx. Now, we will use the fact that f of negative y, evaluated z minus x, is simply equal to f of y evaluated at x minus z.

To see why this is true, let's consider, let's say, a discreet random variable, y. And now, the probability that negative y is equal to negative 1 is simply the same as probability that y is equal to 1. And the same is true for probability density functions. With this fact in mind, we can further write equality as the integration of x times f of y x minus z dx.

We're now ready to compute. We'll first look at the case where z is less than 0. On the right, I'm writing out the distribution of an exponential random variable with a parameter lambda. In this case, using the integration above, we could write it as 0 to infinity, lambda e to the negative lambda x times lambda e to the negative lambda x minus z dx.

Now, the reason we chose a region to integrate from 0 to positive infinity is because anywhere else, as we can verify from the expression of fx right here, that the product of fx times fy here is 0. Follow this through.

We'll pull out the constant. Lambda e to the lambda z, the integral from 0 to infinity, lambda e to the negative 2 lambda x dx. This will give us lambda e to the lambda z minus 1/2 e to the negative 2 lambda x infinity minus this expression value at 0.

And this will give us lamdba over 2 e to the lambda z. So now, we have an expression for f of z evaluated at little z when little z is less than 0. Now that have the distribution of f of z when z is less than 0, we'd like to know what happens when z is greater or equal to 0. In principle, we can go through the same procedure of integration and calculate that value.

But it turns out, there's something much simpler. z is the difference between x and y, at negative z, simply the difference between y and x. Now, x and y are independent and identically distributed. And therefore, x minus y has the same distribution as y minus x.

So that tells us z and negative z have the same distribution. What that means is, is the distribution of z now must be symmetric around 0. In other words, if we know that the shape of f of z below 0 is something like that, then the shape of it above 0 must be symmetric. So here's the origin.

For example, if we were to evaluate f of z at 1, well, this will be equal to the value of f of z at negative 1. So this will equal to f of z at negative 1. Well, with this information in mind, we know that in general, f of z little z is equal to f of z negative little z. So what this allows us to do is to get all the information for z less than 0 and generalize it to the case where z is greater or equal to 0.

In particular, by the symmetry here, we can write, for the case z greater or equal to 0, as lambda over 2 e to the negative lambda z. So the negative sign comes from the fact that the distribution of f of z is symmetric around 0. And simply, we can go back to the expression here to get the value. And all in all, this implies that f of z little z is equal to lambda over 2 e to the negative lambda absolute value of z.

This completes our problem.