Quiz 1 Solutions:
October 13, 2009

1. **(10 points)** We start first by listing the following probabilities:

$$\begin{aligned}
\mathbf{P}(\text{gala}) &= p \\
\mathbf{P}(\text{honey crisp}) &= 1-p \\
\mathbf{P}(\text{ripe} \mid \text{gala}) &= g \\
\mathbf{P}(\text{ripe} \mid \text{honey crisp}) &= h.
\end{aligned}$$

The probability that Bob ate a ripe gala is:

$$\begin{aligned}
\mathbf{P}(\text{ripe gala}) &= \mathbf{P}(\text{gala})\mathbf{P}(\text{ripe} \mid \text{gala}) \\
&= pg.
\end{aligned}$$

2. (a) **(10 points)** If K is the number of gala apples selected from the n apples, then K is a binomial random variable. The probability that $K = k$ is:

$$p_K(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise.} \end{cases}$$

(b) i. **(12 points)** Among the $n+1$ apples, his bounty includes: 1 ripe gala from Caleb, k gala, $n-k$ honey crisp. We can compute the probability of eating a ripe apple conditioned on Bob selecting an apple from each of the three disjoint cases listed above. The total probability theorem allows us to then find the probability of Bob eating a ripe apple.

$$\begin{aligned}
\mathbf{P}(\text{ripe apple}) &= \mathbf{P}(\text{ripe apple} \mid \text{Caleb's ripe gala})\mathbf{P}(\text{Caleb's ripe gala}) \\
&\quad + \mathbf{P}(\text{ripe apple} \mid k \text{ gala})\mathbf{P}(k \text{ gala}) \\
&\quad + \mathbf{P}(\text{ripe apple} \mid n-k \text{ honey crisp})\mathbf{P}(n-k \text{ honey crisp}) \\
&= 1 \cdot \frac{1}{n+1} + g \cdot \frac{k}{n+1} + h \cdot \frac{n-k}{n+1} \\
&= \frac{1 + gk + h(n-k)}{n+1}.
\end{aligned}$$

ii. **(12 points)** Using Bayes' Rule:

$$\mathbf{P}(\text{gala} \mid \text{ripe apple}) = \frac{\mathbf{P}(\text{ripe} \cap \text{gala})}{\mathbf{P}(\text{ripe apple})}.$$

We can compute the $\mathbf{P}(\text{ripe} \cap \text{gala})$ as such:

$$\begin{aligned}
\mathbf{P}(\text{ripe} \cap \text{gala}) &= \mathbf{P}(\text{ripe} \cap \text{gala} \mid \text{Caleb's ripe gala})\mathbf{P}(\text{Caleb's ripe gala}) \\
&\quad + \mathbf{P}(\text{ripe} \cap \text{gala} \mid k \text{ gala})\mathbf{P}(k \text{ gala}) \\
&\quad + \mathbf{P}(\text{ripe} \cap \text{gala} \mid n-k \text{ honey crisp})\mathbf{P}(n-k \text{ honey crisp}) \\
&= 1 \cdot \frac{1}{n+1} + g \cdot \frac{k}{n+1} + 0 \cdot \frac{n-k}{n+1} \\
&= \frac{1+gk}{n+1}.
\end{aligned}$$

Combining this result with that of (i),

$$\mathbf{P}(\text{gala} \mid \text{ripe apple}) = \frac{\frac{1+gk}{n+1}}{\frac{1+gk+h(n-k)}{n+1}} = \frac{1+gk}{1+gk+h(n-k)}.$$

(c) **(10 points)** Let A be the event that the first 10 apples picked were all gala and let B be the event that exactly 10 gala apples were picked out of the 20 apples.

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}.$$

The probability of the event B can be computed by using the result in (a), where $n = 20$ and $k = 10$. $\mathbf{P}(B) = \binom{20}{10} p^{10}(1-p)^{10}$. $\mathbf{P}(A \cap B)$ can be computed by seperating the event of A and B into disjoint events. The event $\{A \cap B\} = \{$1st 10 apples picked are gala and the last 10 apples are honey crisp$\}$. $\mathbf{P}(A \cap B) = p^{10}(1-p)^{10}$. Therefore,

$$\mathbf{P}(A \mid B) = \frac{p^{10}(1-p)^{10}}{\binom{20}{10} p^{10}(1-p)^{10}} = \frac{1}{\binom{20}{10}}.$$

3. (a) **(10 points)** Bob picks apples from each tree until he finds one that is not ripe, and so if we think of the event "picking an unripe apple" as a "success", then X_i is the number of apples picked (i.e. trials) until we get our first success. Therefore X_i is a geometrically distributed random variable with probability of success $1-g$.

Since each tree has a random collection of apples, the apples Bob picks from the first tree are independent of the apples he picks from the second tree, and therefore X_1 and X_2 are independent and identically distributed random variables.

The PMF of X_i is

$$\begin{aligned}
p_{X_1}(k) = p_{X_2}(k) &= (1-(1-g))^{k-1}(1-g) \\
&= \begin{cases} g^{k-1}(1-g), & k = 1, 2, 3, \ldots, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

the expectation of X_i is

$$\mathbf{E}[X_1] = \mathbf{E}[X_2] = \frac{1}{1-g},$$

and the variance of X_i is

$$\begin{aligned}
\text{var}(X_1) = \text{var}(X_2) &= \frac{1-(1-g)}{(1-g)^2} \\
&= \frac{g}{(1-g)^2}.
\end{aligned}$$

(b) **(12 points)** Given that $Y_2 = (X_1 - 1) + (X_2 - 1)$, we can use the linearity property of expectations to find the expectation of Y_2.

$$\begin{aligned}
\mathbf{E}[Y_2] &= \mathbf{E}[(X_1 - 1) + (X_2 - 1)] \\
&= \mathbf{E}[X_1] + \mathbf{E}[X_2] - 2 \\
&= \frac{2}{1-g} - 2 \\
&= \frac{2g}{1-g}.
\end{aligned}$$

Since adding or subtracting a constant from a random variable has no effect on its variance, and since X_1 and X_2 are independent, we have

$$\begin{aligned}
\text{var}(Y_2) &= \text{var}((X_1 - 1) + (X_2 - 1)) \\
&= \text{var}(X_1 + X_2) \\
&= \text{var}(X_1) + \text{var}(X_2) \\
&= \frac{2g}{(1-g)^2}.
\end{aligned}$$

(c) **(12 points)**
Let $k \geq 0$ and $\ell \geq k$. The event "$Y_1 = k$ and $Y_2 = \ell$" is identical to the event "$X_1 = k + 1$ and $X_2 = \ell - k + 1$". Since X_1 and X_2 are independent, the desired probability is $(1-g)g^{k+1-1}(1-g)g^{\ell-k+1-1} = (1-g)^2 g^{\ell}$, when $0 \leq k \leq \ell$, and zero otherwise.

$$p_{Y_1,Y_2}(k,\ell) = \begin{cases} g^{\ell}(1-g)^2, & k = 0, 1, \ldots \ell, \text{ and } \ell = k, k+1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Note that we can interpret this result as the probability of ℓ failures (i.e. picking ℓ ripe apples) and two successes (i.e. the two unripe apples that cause Bob to stop at each tree). Note also that although the expression for the joint PMF doesn't depend on k, the sample space does depend on k because $\ell \geq k$: Bob cannot pick more ripe apples in the first tree than he does in the first two trees combined.

(d) (i) **(5 points)** No. X_1 and Y_2 are not independent because $Y_2 \geq X_1 - 1$ and so knowing X_1 gives us information about Y_2. Specifically, if we know that Bob picked 10 total apples from the first tree (i.e. $X_1 = 10$), then we also know that Bob must have picked at least 9 ripe apples from the first two trees combined (i.e. $Y_2 \geq 9$).

(ii) **(5 points)** Yes. X_2 relates only to the second tree and Y_1 relates only to the first tree, and since the picking of apples is independent across trees, the two random variables are independent.