Mathematics for Computer Science

MIT 6.042J/18.062J

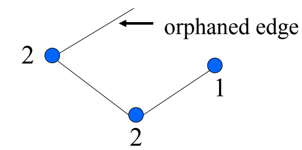
Graphs:

Handshaking, Connectivity & Trees

Possible Graph?

Is there a graph with vertex degrees 2,2,1?

NO!



An Impossible Graph

2+2+1 = odd

The Handshaking Lemma:

The sum of degrees must be even.

$$2|E| = \sum_{v \in V} \deg(v)$$

The Handshaking Lemma

Proof: Each edge contributes 2 to the sum on the right

$$2|E| = \sum_{v \in V} \deg(v)$$

Paths & Simple Paths

Path: $v_0, v_1, \ldots, v_n$ for $n \geq 0$

where $v_i - - v_{i+1}$ is an edge for all i s.t. $0 \leq i < n$

Simple path: all v_i different

Connectedness

Vertices *v*, *w* are connected iff there is a path starting at *v* and ending at *w*.

A graph is connected iff every two of its vertices are connected

Paths & Simple Paths

Lemma: The shortest path connecting two vertices is simple!

Simple path: all v_i different

Cycles & Simple Cycles

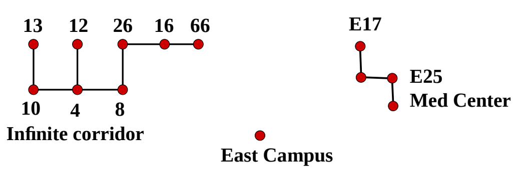
Cycle: $v_0, v_1, \ldots, v_n, v_0$ for $n \geq 1$
where $v_i \text{ --- } v_{i+1}$ is an edge for all i s.t. $0 \leq i < n$
and $v_n \text{ --- } v_0$

Simple cycle: all v_i different and $n \geq 2$

same cycle: $v_2, v_1, v_0, v_n, v_{n-1}, \ldots, v_3, v_2$

Connected Components

The more connected components, the more broken up the graph is.



Connected Components

The connected component of vertex v is

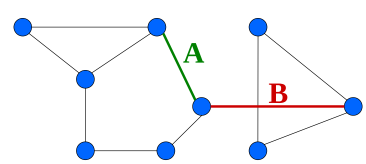
$$\{w \mid v \text{ and } w \text{ are connected}\}$$

Same as the block $[v]$ of the *connectedness relation*

Cut Edges

An edge is a *cut edge* if removing it from the graph disconnects two vertices.

Cut Edges



A is not, *B* is

Cut Edges and Cycles

Lemma: An edge is a cut edge *iff* it is not part of a simple cycle.

Proof: Class problem.

Cut Edges

Fault-tolerant design:

In a tree, every edge is a cut edge (bad)
In a mesh, no edge is a cut edge (good)
Tradeoff edges for failure tolerance

k-Connectedness

Def: *k*-connected iff remains connected when any *k*-1 edges are deleted.

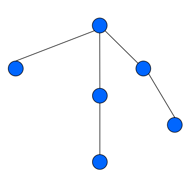
k-Connectedness

Example:

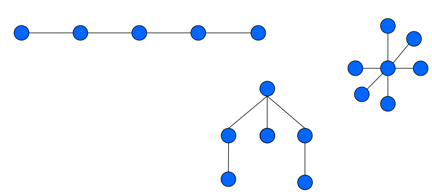
K_n is (n-1)-connected

Trees

A ***tree*** is a connected graph with no cycles.



More Trees



Other Tree Definitions

Lemma: A tree is a connected graph with n vertices and $n-1$ edges.

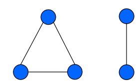
Lemma: A tree is an edge-minimal connected graph on a set of vertices.

Lemma: A tree is a graph with a **unique** path between any 2 vertices.

Be careful with these definitions

A tree is a graph with n vertices and n – 1 edges??

NO:



Spanning Tree

Def: A subgraph that is a tree with all the vertices.

Always exists: any minimum-edge-size, connected subgraph on all the vertices

Problems

4 & 5