Solutions to In-Class Problems Week 4, Fri.

Problem 1. **(a)** For any vertex, v, in a graph, let $\widehat{v}$ be the set of vertices adjacent to v, that is,

$$\widehat{v} ::= \left\{ v' \mid v\text{—}v' \text{ is an edge of the graph} \right\}.$$

Suppose f is an isomorphism from graph G to graph H. Carefully prove that $f(\widehat{v}) = \widehat{f(v)}$.

Solution. We first show that $\widehat{f(v)} \subseteq f(\widehat{v})$ by showing that if $w \in \widehat{f(v)}$, then $w \in f(\widehat{v})$.

Now $w \in \widehat{f(v)}$ means that $w\text{—}f(v)$ is an edge of H. Since f is an isomorphism, there must be some v' such that $w = f(v')$. So $f(v')\text{—}f(v)$ is an edge of H, and therefore $v'\text{—}v$ is an edge of G, by definition of isomorphism. This means $v' \in \widehat{v}$, and so $f(v') \in f(\widehat{v})$ by definition of $f(\widehat{v})$. So $w = f(v') \in f(\widehat{v})$, as required.

Conversely, we show that $f(\widehat{v}) \subseteq \widehat{f(v)}$ by showing that if $w \in f(\widehat{v})$, then $w \in \widehat{f(v)}$.

But $w \in f(\widehat{v})$ means that $w = f(v')$ for some v' adjacent to v in G. This means $v\text{—}v'$ is an edge of G, and so $f(v)\text{—}f(v')$ is an edge of H by definition of isomorphism. So $w = f(v')$ is adjacent to $f(v)$; in other words, $w \in \widehat{f(v)}$, as required. ■

(b) Conclude that if G and H are isomorphic graphs, then for each $k \in \mathbb{N}$, they have the same number of degree k vertices.

Solution. By definition, $\deg(v) = |\widehat{v}|$. Since an isomorphism is a bijection, a set and its image will be the same size (by the Mapping Rule from Week 2 Notes), so the Lemma of part (a) implies that an isomorphism, f, maps degree k vertices to degree k vertices. This means that the image under f of the set of degree k vertices of G is precisely the set of degree k vertices of H. So by the Mapping Rule again, there are the same number of degree k vertices in G and H. ■

Problem 2. For each of the following pairs of graphs, either define an isomomorphism between them, or prove that there is none. (We write ab as shorthand for $a\text{—}b$.)

(a)

$$G_1 \text{ with } V_1 = \{1, 2, 3, 4, 5, 6\}\,,\ E_1 = \{12, 23, 34, 14, 15, 35, 45\}$$
$$G_2 \text{ with } V_2 = \{1, 2, 3, 4, 5, 6\}\,,\ E_2 = \{12, 23, 34, 45, 51, 24, 25\}$$

Solution. Not isomorphic: G_2 has a node, 2, of degree 4, but the maximum degree in G_1 is 3. ■

(b)

$$G_1 \text{ with } V_1 = \{1, 2, 3, 4, 5, 6\}\,,\ E_1 = \{12, 23, 34, 14, 45, 56, 26\}$$
$$G_2 \text{ with } V_2 = \{a, b, c, d, e, f\}\,,\ E_2 = \{ab, bc, cd, de, ae, ef, cf\}$$

Solution. Isomorphic with the vertex correspondence: $1f, 2c, 3d, 4e, 5a, 6b$ ■

(c)

$$G_1 \text{ with } V_1 = \{a, b, c, d, e, f, g, h\}\,,\ E_1 = \{ab, bc, cd, ad, ef, fg, gh, he, dh, bf\}$$
$$G_2 \text{ with } V_2 = \{s, t, u, v, w, x, y, z\}\,,\ E_2 = \{st, tu, uv, sv, wx, xy, yz, wz, sw, vz\}$$

Solution. Not isomorphic: they have the same number of vertices, edges, and set of vertex degrees. But the degree 2 vertices of G_1 are all adjacent to two degree 3 vertices, while the degree 2 vertices of G_2 are all adjacent to one degree 2 vertex and one degree 3 vertex. ■

Problem 3. Extra Problem.

(a) Exhibit three nonisomorphic, connected graphs with five vertices and four edges.

(b) Argue that every connected graph with five vertices and four edges is isomomorphic to one of the three in part (a).