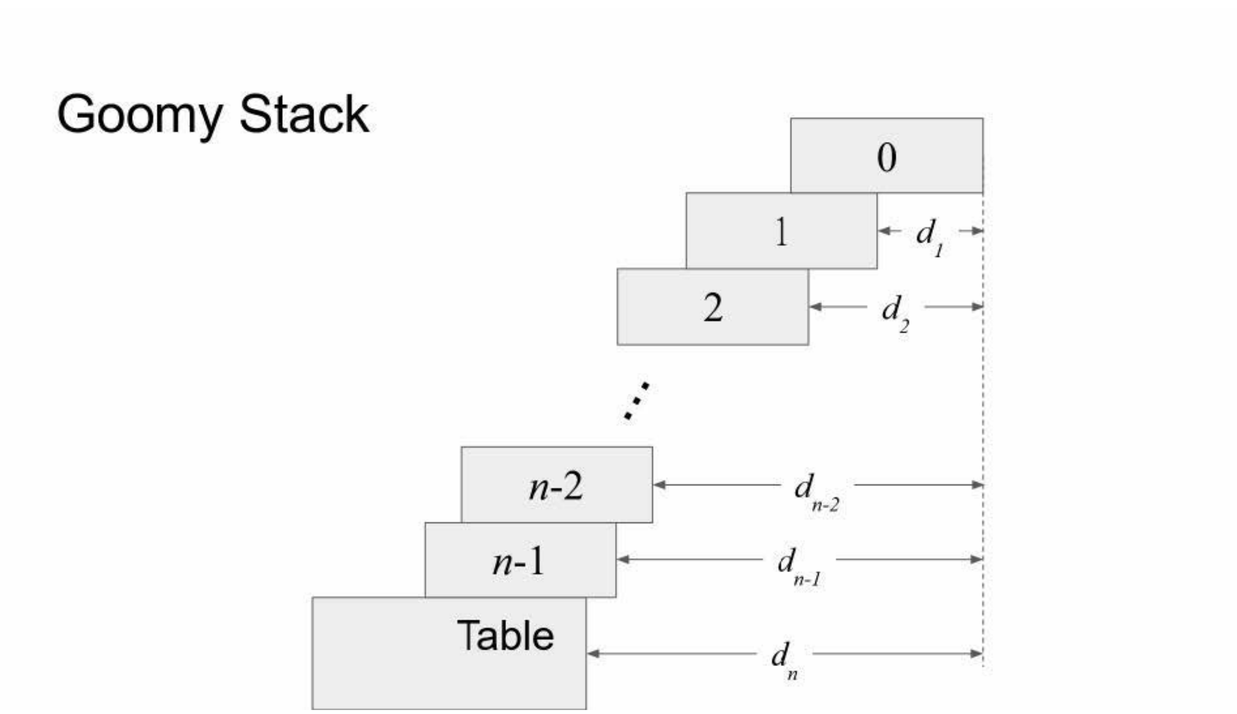
Lecture 06: Asymptotics

1 Goomy Stack

Last week we saw some techniques for evaluating and approximating sums. Today we will use those techniques to solve a well-known physics problem.

1.1 Setup



The problem setup is illustrated above.

Rules:

1. We have a stack of n Goomy, numbered 0 through $n-1$. We consider the table to be Goomy n, for ease of notation.
2. For every i, Goomy i must be sitting on Goomy $i+1$ (no two Goomy can be at the same height).

3. Some Goomy must reach 1 foot (Goomy's width) past the edge of the table.

Is this possible? Or is there some fundamental physical law that says the stack must fall over if one Goomy is that far out?

1.2 Solution

We will see that in fact, it is possible. Define d_i to be the horizontal distance between the right edge of Goomy i and the right edge of Goomy 0. If we can set all of the d_i so that the stack is stable, and $d_n > 1$, then this puts the top Goomy 1 foot past the edge of the table.

The d_i are subject to constraints given by gravity. Note that if the top Goomy is more than half off the next Goomy, its center of mass is not above its support, so it will fall off the stack. Similarly, if for any k, the top k Goomy together don't have their collective center of mass above the next Goomy, then they will all fall off. In other words, are constraints are that for every k, the center of mass of the top k Goomy together must be above the next Goomy.

Notice that the right edge of Goomy i is at location d_i, and its center is another $1/2$ foot beyond its right edge. Therefore, the center of mass of Goomy i is at location $d_i + 1/2$.

The top k Goomy are Goomy 0, Goomy 1, $\ldots$, Goomy $k-1$, so their center of mass collectively is

$$\frac{1}{k}\left(d_0 + 1/2 + d_1 + 1/2 + \cdots + d_{k-1} + 1/2\right) = \frac{1}{2} + \frac{1}{k}\sum_{i=0}^{k-1} d_i.$$

The next Goomy is Goomy k, which goes from location d_k to $d_k + 1$, so we have

$$d_k \leq \frac{1}{2} + \frac{1}{k}\sum_{i=0}^{k-1} d_i \leq d_k + 1.$$

In particular, if we set

$$d_k = \frac{1}{2} + \frac{1}{k}\sum_{i=0}^{k-1} d_i,$$

or equivalently

$$k \cdot d_k = \frac{k}{2} + \sum_{i=0}^{k-1} d_i,$$

(i.e. the COM of the top k Goomy collectively is *exactly* at the right edge of the next Goomy), then this will satisfy both inequalities. This is the *greedy* approach; we construct our stack from the top down, and we always put the top k Goomy as far to the right as they can go without falling off the next Goomy.

We now have a recurrence for d_k. Note that the recurrence is satisfied for *every* k, so we can use the perturbation method to simplify our recurrence. If we write the recurrence for both d_k and d_{k-1} and subtract the two, we have

$$\begin{aligned} k \cdot d_k &= \frac{k}{2} + \sum_{i=0}^{k-1} d_i \\ (k-1) \cdot d_{k-1} &= \frac{k-1}{2} + \sum_{i=0}^{k-2} d_i \\ k \cdot d_k - (k-1) \cdot d_{k-1} &= \frac{1}{2} + d_{k-1} \end{aligned}$$

We can rearrange a little bit to get

$$d_k = d_{k-1} + \frac{1}{2k}.$$

We now have a much simpler recurrence for d_k which only depends on d_{k-1} (as opposed to all of $d_0, d_1, \ldots, d_{k-1}$), and if we expand out d_n, we have

$$d_n = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i}.$$

2 Harmonic Numbers

Definition 1 (Harmonic Numbers)**.** *The nth Harmonic Number is* $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

Using this definition, we have $d_n = \frac{1}{2} H_n$. If we compute the first few harmonic numbers, we get $H_1 = 1$, $H_2 = \frac{3}{2}$, $H_3 = \frac{11}{6}$, and $H_4 = \frac{25}{12} > 2$. Since $H_4 > 2$, this means that $d_4 > 1$, so with only four Goomy, we can get one of them to extend a foot past the edge of the table, as we wanted.

We know that the harmonic numbers diverge to infinity, so in principle, by making the Goomy stack tall enough, we could extend as far as we like past the edge of the table. For instance, $H_{227} \approx 6.004$, so with 227 Goomy, we could go 1 yard (3 feet) past the edge of the table.

So we know that it is possible to go as far as we like, but a natural question is: How many Goomy do we need? More formally, one could ask: Given x, what is the minimal n such that $H_n \geq x$? Can we compute this n in a closed form as a function of x?

Unfortunately, this would entail computing H_n, and despite many brilliant mathematicians trying to find an answer for many years, nobody knows how to compute H_n in a closed form. However, we can approximate H_n! Furthermore, with a good enough approximation for H_n, we can also approximate the minimal n such that $H_n \geq x$.

In lecture 05, we saw a way to approximate a sum using an integral. In this case, the terms in our sum are decreasing, so we'll use the integral bound for the sum of a decreasing sequence.

$$f(n) + \int_1^n f(x)\,\mathrm{d}x \leq \sum_{i=1}^{n} f(i) \leq f(1) + \int_1^n f(x)\,\mathrm{d}x$$

If we put in $f(x) = \frac{1}{x}$, then this turns into:

$$\begin{aligned}
\frac{1}{n} + \int_1^n \frac{\mathrm{d}x}{x} \quad &\leq H_n \leq \quad 1 + \int_1^n \frac{\mathrm{d}x}{x} \\
\frac{1}{n} + [\ln x]_1^n \quad &\leq H_n \leq \quad 1 + [\ln x]_1^n \\
\frac{1}{n} + \ln n - \ln 1 \quad &\leq H_n \leq \quad 1 + \ln n - \ln 1 \\
\frac{1}{n} + \ln n \quad &\leq H_n \leq \quad 1 + \ln n
\end{aligned}$$

We now have an upper bound and a lower bound that differ from each other (and hence from the true value of H_n) by less than 1! In fact, if we just use $\ln n$ as an approximation for H_n, then the error is at most 1. As n goes to infinity, $\ln n$ also goes to infinity, so the error term of 1 becomes negligible in comparison. We ignore this error term and say that $H_n \sim \ln n$.

Definition 2 (Tilde notation)**.** *$f \sim g$ (read "f tilde g" or "f is tilde of g") if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Tilde notation gives a sense in which f and g are "approximately equal"; if we only care about the approximate limiting behavior, we can safely ignore the precise difference between f and g and treat them the same.

Using tilde notation, we have $H_n \sim \ln n$. With this approximation, we can see that if we want a Goomy to go one chain (22 yards) past the edge of the table, we have $H_n = 2d_n \approx 2 \cdot 22 \cdot 3 = 132$, so $n \approx e^{132}$, or about one octodecillion (10^{57}). Mathematically, it's possible, but physically, we start having issues like having left Earth's gravity well (at about a billion Goomy), stars getting in the way (at about 10^{18} Goomy), the stack being unobservable due to Heisenberg's Uncertainty Principle (at about 10^{36} Goomy), and the stack simply containing more mass than the entire universe (at about 10^{54} Goomy).

3 Approximating Products

In the previous section, we approximated the harmonic numbers; we will now see how to approximate products.

Definition 3 ($\prod$ notation)**.** $\prod_{i=1}^{n} x_i$ *denotes the product* $x_1 \times x_2 \times \cdots \times x_n$.

Just as $\sum$ is used to denote a sum, $\prod$ is used to denote a product. Perhaps the best known product is $n!$.

Definition 4 ($n!$)**.** $n!$ *(read "n factorial") is the product of the smallest* n *positive integers, given by* $n! = \prod_{i=1}^{n} i$.

Like H_n, $n!$ is very important in computer science, particularly in counting and probability. It can be computed iteratively as per the definition, but we would like to compute it much more efficiently. Just as we can compute a^n with repeated squaring using about $\log_2$ multiplications rather than n, we would like to find a similarly efficient algorithm for computing $n!$.

We already know techniques for computing sums, so we can reduce the task of computing a product to computing a sum by using logs. We then have

$$\begin{aligned} \ln(n!) &= \ln\left(\prod_{i=1}^{n} i\right) \\ &= \ln(1 \times 2 \times 3 \times \cdots \times n) \\ &= \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n \\ &= \sum_{i=1}^{n} \ln i \end{aligned}$$

Unfortunately, nobody knows how to compute this sum in a closed form, so rather than compute it exactly, we can approximate it. Similarly to how we handled the harmonic numbers, we use an integration bound. This time, however, we are summing an increasing sequence, so we use the integration bound for increasing sequences.

$$f(1) + \int_1^n f(x)\,\mathrm{d}x \le \sum_{i=1}^{n} f(i) \le f(n) + \int_1^n f(x)\,\mathrm{d}x$$

This time, $f(x) = \ln x$, and we evaluate the integral using integration by parts. Taking

$u = \ln x$ and $v = x$, we have

$$\begin{aligned}
\int_1^n f(x)\,\mathrm{d}x &= \int_1^n \ln x\,\mathrm{d}x \\
&= \int_1^n u\,\mathrm{d}v \\
&= uv\,|_1^n - \int_1^n v\,\mathrm{d}u \\
&= [x\ln x]_1^n - \int_1^n dx \\
&= [x\ln x - x]_1^n \\
&= n\ln n - n - 1\ln 1 + 1 \\
&= n\ln n - n + 1
\end{aligned}$$

Putting this back into our integration bound and exponentiating, we have

$$\begin{array}{rcl}
\ln 1 + n\ln n - n + 1 & \leq \ln(n!) \leq & \ln n + n\ln n - n + 1 \\
n\ln n - n + 1 & \leq \ln(n!) \leq & (n+1)\ln n - n + 1 \\
\dfrac{n^n}{e^{n-1}} & \leq n! \leq & \dfrac{n^{n+1}}{e^{n-1}}
\end{array}$$

This time, we approximated to within a factor of n, rather than to within an additive constant. However, considering how large $n!$ is (bigger than exponential), even a multiplicative factor of n is actually quite good! And in fact, $n!$ is so important that many people have derived much more precise approximations.

3.1 Stirling's Formula

One such approximation, known as Stirling's formula, is the following:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{\epsilon(n)},$$

$$\text{where } \frac{1}{12n+1} \leq \epsilon(n) \leq \frac{1}{12n}$$

These are extremely precise; the multiplicative error is less than $1 + \dfrac{1}{144n^2}$ (as opposed to at most n, as we derived above). Since $\epsilon(n) \to 0$ as $n \to \infty$, we can in fact drop the $e^{\epsilon(n)}$ entirely and simply write

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Now we have a very precise approximation for $n!$, and moreover, we can compute it extremely efficiently using repeated squaring.

4 Asymptotic Notation

We just saw two examples of tilde notation being used to compare values we care about (H_n and $n!$) with two values that are much easier to compute and work with. As n grows, we don't really care about the precise difference between the approximation and the true value, and so we treat them as essentially "equal". In computer science, we can often get away with *much less precise* approximations. For instance, when talking about the runtime of an algorithm, we don't really even care about multiplicative constant factor errors. Machines can differ, and so we want to normalize out such factors as how long a single instruction takes to run on a particular machine. What we really care about is how the runtime of our algorithm changes as the input size continues to grow. In order to talk about such approximations, we use *asymptotic notation*. Asymptotic notation talks about upper and lower bounds on functions, modulo constant factors and bounded, finite exceptions. While somewhat complicated to define, it captures an intuitive concept of approximation that will allow you to reason about the "big picture" behavior of functions without getting bogged down with minor details.

In all of the following, assume that $g : \mathbb{Z}^+ \to \mathbb{R}^+$, and $f : \mathbb{Z}^+ \to \mathbb{R}$, but note that all definitions extend naturally to other domains.

Reminder: Our first example of asymptotic notation was $\sim$.

Definition 5 (Tilde)**.** *$f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.*

4.1 Big-O

Our next example, and the most commonly used in computer science is Big-O notation.

Definition 6 (Big-O)**.** *$f \in O(g)$ (read "f is in Big-O of g") iff*

$$\exists c \in \mathbb{R}.\, \exists M \in \mathbb{Z}^+.\, \forall x \in \mathbb{Z}^+.\, \left[x > M \Rightarrow |f(x)| \le c \cdot g(x)\right].$$

$O(g)$ is a set of functions that are *asymptotically upper bounded* by g. This definition is somewhat complicated, so let's unpack it. Essentially, it is expressing that (approximately) $|f| \le g$. The c value means that we don't really care about constant factors, so f just needs to be within some constant factor of being bounded by g. The M value means that we also don't care about the behavior of f on some bounded, finite set close to 0. As long as x is sufficiently large, f exhibits the behavior we care about (being within a constant factor of g). We think of Big-O notation as capturing a notion of $\le$, modulo constant factors and bounded exceptions. Now this definition looks a little bit like the ϵ-δ definitions of limits from calculus, but it is *NOT* a limit! That said, we do have the following:

Theorem 1. *If $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} \in \mathbb{R}$, then $f \in O(g)$.*

Proof. Suppose $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = l \in \mathbb{R}$. Take $c = l + 1$. From the definition of a limit, $\exists M$ s.t. $\forall x > M$, $\left|\frac{|f(x)|}{g(x)} - l\right| < 1$, i.e. $l - 1 < \frac{|f(x)|}{g(x)} < l + 1 = c$. Hence $f \in O(g)$. □

Remember that this limit test can be useful for proving that $f \in O(g)$. However, the converse does not hold; if the limit does not exist, then you must go back to the definition of $O(g)$.

Examples

1. If $f(x) = x$ and $g(x) = x^2$, then $f \in O(g)$.
   Proof: Limit is 0.
2. If $f(x) = 3\sin x$, and $g(x) = 1$, then $f \in O(g)$.
   Proof: Take $M = 0$ and $c = 3$.
3. If $f(x) = x^2$ and $g(x) = x$, then $f \notin O(g)$.
   Proof: For any c and M, we can take $x > \max(c, M)$. Then $x^2 > c \cdot x$, so $f \notin O(g)$.
4. If f is quadratic, and $g(x) = x^2$, then $f \in O(g)$.
   Proof: Limit is leading coefficient of f.
5. If f is a polynomial, and $g(x) = 2^x$, then $f \in O(g)$.
   Proof: Limit is 0.
6. If $f(x) = 4^x$, and $g(x) = 2^x$, then $f \notin O(g)$.
   Proof: For any c and M, we can take $x > \max(c, M)$. Then $4^x > x \cdot 2^x > c \cdot 2^x$, so $f \notin O(g)$.

You may also see the following simpler, but equivalent, characterization of Big-O.

Theorem 2. *$f \in O(g)$ iff*

$$\exists c' \in \mathbb{R}. \forall x \in \mathbb{Z}^+. |f(x)| \le c' \cdot g(x).$$

Proof. For the reverse direction, take $M = 0$ and $c' = c$. Then for all $x > M$, $|f(x)| \le c \cdot g(x) = c' \cdot g(x)$, so $f \in O(g)$. For the forward direction, assume $f \in O(g)$. Let $S = \left\{\frac{|f(x)|}{g(x)} : x \le M\right\} \cup \{c\}$, and take $c' = \max(S)$. Now let $x \in \mathbb{Z}^+$. If $x \le M$, then $|f(x)| = \frac{|f(x)|}{g(x)} \cdot g(x) \le c' \cdot g(x)$. Otherwise, $|f(x)| \le c \cdot g(x) \le c' \cdot g(x)$. Either way, $|f(x)| \le c' \cdot g(x)$. □

Note that in the above, we rely on S being a finite set. This theorem is valid when g's domain is $\mathbb{Z}^+$ (or $\mathbb{N}$ or some other discrete set), but NOT if we extend the definition of Big-O to e.g. $\mathbb{R}^+$. When f and g have a discrete domain, you may use Theorem 2 as your definition of Big-O, but it will not generalize to all domains.

4.2 Little-o

Definition 7 (Little-o)**.** *$f \in o(g)$ (read "f is in little-o of g") if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

$o(g)$ is the set of functions that are *asymptotically much smaller* than g. If we think of Big-O as representing the non-strict inequality $\leq$, then little-o is its strict counterpart, capturing a notion of $<$. Like tilde and unlike Big-O, little-o does use a limit as its definition. Of course, you may also unpack the definition of a limit and use the following equivalent characterization as a definition.

Theorem 3. *$f \in o(g)$ iff*

$$\forall \epsilon \in \mathbb{R}^+ .\, \exists M \in \mathbb{Z}^+ .\, \forall x \in \mathbb{Z}^+ .\ [x > M \Rightarrow |f(x)| < \epsilon \cdot g(x)] \,.$$

Proof. Definition of limit. □

Examples

1. If $f(x) = x$ and $g(x) = x^2$, then $f \in o(g)$.
   Proof: Limit is 0.
2. If $f(x) = 3\sin x$, and $g(x) = 1$, then $f \notin o(g)$.
   Proof: Limit doesn't exist.
3. If $f(x) = x^2$ and $g(x) = x$, then $f \notin o(g)$.
   Proof: Limit is ∞.
4. If f is quadratic, and $g(x) = x^2$, then $f \notin o(g)$.
   Proof: Limit is leading coefficient of f (non-zero).
5. If f is a polynomial, and $g(x) = 2^x$, then $f \in o(g)$.
   Proof: Limit is 0.
6. If $f(x) = 4^x$, and $g(x) = 2^x$, then $f \notin o(g)$.
   Proof: Limit is ∞.

Just as $x < y \Rightarrow x \leq y$, we also have the following:

Theorem 4. *If $f \in o(g)$, then $f \in O(g)$.*

Proof. Follows immediately from Theorem 1. □

4.3 Big-Ω

Definition 8 (Big-Ω)**.** *$f \in \Omega(g)$ (read "f is in Big-Omega of g") if $g \in O(f)$. (Note that this implies f is positive.)*

$\Omega(g)$ is the set of functions that are *asymptotically lower bounded* by g. If we think of Big-O as representing the non-strict inequality $\leq$, then Big-Ω represents the non-strict inequality $\geq$. Just as with Big-O, we have the following limit test:

Theorem 5. *If $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} \in (0, \infty]$, then $f \in \Omega(g)$.*

Proof. Follows from Theorem 1. □

Examples

1. $x^2 \in \Omega(x)$
2. $2^x \in \Omega(x^2)$
3. $\frac{x}{100} \in \Omega(100x + \sqrt{x})$

Theorem 6. *If $f \in o(g)$, then $f \notin \Omega(g)$.*

Proof. Suppose for sake of contradiction that $f \in o(g)$ and $g \in O(f)$, i.e.

$$\forall \epsilon \in \mathbb{R}^+. \exists M_\epsilon \in \mathbb{Z}^+. \forall x \in \mathbb{Z}^+. \; [x > M_\epsilon \Rightarrow f(x) < \epsilon \cdot g(x)]$$

and

$$\exists c \in \mathbb{R}^+. \forall x \in \mathbb{Z}^+. \, g(x) \leq c \cdot f(x).$$

Taking $\epsilon = \frac{1}{c}$ and $x = M_\epsilon + 1$ gives $g(x) \leq c \cdot f(x) < g(x)$, a contradiction. □

4.4 Little-ω

Definition 9 (Little-ω)**.** *$f \in \omega(g)$ (read "f is in little-omega of g") if $g \in o(f)$. (Note that this implies f is positive.)*

$\omega(g)$ is the set of functions that are *asymptotically much larger* than g. As with our earlier analogies, little-ω represents the strict inequality $>$. Just as with little-o, we have equivalent characterizations, any of which may be used as a definition.

Theorem 7. *The following are equivalent:*

1. $f \in \omega(g)$

2. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

3. $\forall c \in \mathbb{R}^+ . \exists M \in \mathbb{Z}^+ . \forall x \in \mathbb{Z}^+ . \; [x > M \Rightarrow f(x) > c \cdot g(x)]$

Proof. Equivalence of (2) and (3) is the definition of the limit. Equivalence of (1) and (3) is Theorem 3, with the additional observation that both f and g are positive. □

Examples

1. $x^2 \in \omega(x)$
2. $2^x \in \omega(x^2)$
3. $\frac{x}{100} \notin \omega(100x + \sqrt{x})$

4.5 Θ

Definition 10. *$f \in \Theta(g)$ (read "f is in Theta of g") if $f \in O(g)$ and $f \in \Omega(g)$. (Note that this implies f is positive.)*

$\Theta(g)$ is the set of functions that are *asymptotically equal* to g. If the previous symbols represent inequalities, then Θ represents $=$. As with Big-O and Big-Ω, there is a limit test which we can use to prove that $f \in \Theta(g)$.

Theorem 8. *If $\lim_{x\to\infty} \frac{f(x)}{g(x)} \in \mathbb{R}^+$, then $f \in \Theta(g)$.*

Proof. Follows from Theorems 1 and 5. □

Examples

1. $10x^3 + 20x^2 + 5 \in \Theta(x^3)$
2. $2 + \sin x \in \Theta(1)$
3. $\frac{x}{\ln x} \notin \Theta(x)$
4. $1 + \sin x \notin \Theta(1)$

4.6 CAUTION

There are many ways to read $f \in O(g)$ (and all of the other analogous statements).

- "f *is in* (Big-) O of g"
- "f *is* (Big-) O of g"
- "f *equals* (Big-) O of g"
- "f *is less than or equal to* (Big-) O of g"

All of these are commonly used, and unfortunately, they invite some wrong ways to write $f \in O(g)$. Many people write $f = O(g)$ or $f \leq O(g)$.

NEVER WRITE $f = O(g)$. Not only is this an abuse of notation, it invites nonsense deductions such as $f = O(g)$ and $h = O(g)$, so $f = h$. Often the fallacies are more subtle, particularly when trying to use different kinds of asymptotic notation simultaneously or manipulate them as sets. Big-O does NOT behave as equality. Writing $f \leq O(g)$ is more acceptable. Although it is an abuse of notation, Big-O does behave somewhat like an inequality. It is also harder to misinterpret $f \leq O(g)$ and write complete garbage. You should still be careful though; Big-O does not obey all of the rules that govern inequalities (Lecture 15).

A second abuse of notation (which I have already been using in these notes) is writing $f(n) \in O(g(n))$, e.g. $n \in O(n^2)$. While not technically correct according to the definitions given (pedantically it should be $(n \mapsto n) \in O(n \mapsto n^2)$), this latter is terrible to read. $n \in O(n^2)$ is clear and unambiguous, so please use this notation in preference to $\mapsto$.

A third abuse of notation which you will likely encounter is statements of the form $f \in g(O(h))$. This does NOT mean $f \in O(g \circ h)$; $g(O(h))$ is shorthand for the set $\{g \circ h' \mid h' \in O(h)\}$. Sometimes these two will coincide, but often they will not.

Another common mistake is writing statements such as $f \geq O(g)$. This means absolutely nothing, because the constant 0 function is in $O(g)$ (so this is just saying that the absolute value of f is non-negative). Remember that O and o are ONLY upper bounds, and Ω and ω are ONLY lower bounds.

Finally, the definitions of Ω and ω presented above are the ones used in computer science, but there are conflicting definitions used in analytic number theory. If you take a course in analytic number theory, you may encounter the definition $f \in \Omega(g)$ iff $f \notin o(g)$, or the corresponding definition $f \in \omega(g)$ iff $f \notin O(g)$. For most of the functions that you will encounter in CS, these definitions will coincide, but they are not generally equivalent (the CS definitions are stronger).

4.7 Summary

	Definition	Limit Test	
$f \sim g$	$(\rightarrow\rightarrow\rightarrow)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$	"$=$"
$f \in O(g)$	$\exists c. \exists M. \forall x > M. \|f(x)\| \leq c \cdot g(x)$	$\lim_{x\to\infty} \frac{\|f(x)\|}{g(x)} \in \mathbb{R}$	"$\leq$"
$f \in o(g)$	$(\rightarrow\rightarrow\rightarrow)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$	"$<$"
$f \in \Omega(g)$	$g \in O(f)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} \in (0, \infty]$	"$\geq$"
$f \in \omega(g)$	$g \in o(f)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$	"$>$"
$f \in \Theta(g)$	$f \in O(g)$ and $f \in \Omega(g)$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} \in (0, \infty)$	"$=$"

Black limit tests are valid definitions; red limit tests are *only one-sided* tests!