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by A. Megretski

Lecture 11: Volume Evolution And System Analysis[1]

Lyapunov analysis, which uses monotonicity of a given function of system state along trajectories of a given dynamical system, is a major tool of nonlinear system analysis. It is possible, however, to use monotonicity of *volumes* of subsets of the state space to predict certain properties of system behavior. This lecture gives an introduction to such methods.

11.1 Formulae for volume evolution

This section presents the standard formulae for evolution of volumes.

11.1.1 Weighted volume

Let U be an open subset of $\mathbf{R}^n$, and $\rho: \ U \mapsto \mathbf{R}$ be a measureable function which is bounded on every compact subset of U. For every hypercube

$$Q(\bar{x}, r) = \{x = [x_1; x_2; \dots; x_n] : \ |x_k - \bar{x}_k| \le r\}$$

contained in U, its *weighted volume* with respect to ρ is defined by

$$V_\rho(Q(\bar{x}, r)) = \int_{\bar{x}_1 - r}^{\bar{x}_1 + r} \left(\int_{\bar{x}_2 - r}^{\bar{x}_2 + r} \left(\dots \left(\int_{\bar{x}_{n-1} - r}^{\bar{x}_{n-1} + r} \left(\int_{\bar{x}_n - r}^{\bar{x}_n + r} \rho(x_1, x_2, \dots, x_n) dx_n \right) dx_{n-1} \right) \dots \right) dx_2 \right) dx_1.$$

Without going into the fine details of the measure theory, let us say that the *weighted volume* of a subset $X \subset U$ with respect to ρ is *well defined* if there exists $M > 0$ such that for every $\epsilon > 0$ there exist (countable) families of cubes $\{Q_k^1\}$ and $\{Q_k^2\}$ (all contained in

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U) such that X is contained in the union of Q_k^i, the union of all Q_k^2 is contained in the union of X and Q_k^1, and

$$\sum_k V_{|\rho|}(Q_k^1) < \epsilon, \quad \sum_k V_{|\rho|}(Q_k^2) < M,$$

in which case the volume $V_\rho(X)$ is (uniquely) defined as the limit of

$$\sum_k V_\rho(Q_k^2)$$

as $\epsilon \longrightarrow 0$ and Q_k^2 are required to have empty pair-wise intersections. A common alternative notation for $V_\rho(X)$ is

$$V_\rho(X) = \int_{x \in X} \rho(x) dx.$$

When $\rho \equiv 1$, we get a definition of the usual (Lebesque) volume. It can be shown that the weighted volume is well defined for every compact subset of U, and also for every open subset of U for which the closure is contained in U. It is important to remember that not every bounded subset of U has a volume, even when $\rho \equiv 1$.

11.1.2 Volume change under a smooth map

The rules for variable change in integration allow one to trace the change of weighted volume under a smooth transformation.

Theorem 11.1 *Let U be an open subset of $\mathbf{R}^n$. Let $F: \mathbf{U} \mapsto \mathbf{U}$ be an injective Lipschitz function which is differentiable on an open subset U_0 of U such that the complement of U_0 in U has zero Lebesque volume. Let $\rho: U \mapsto \mathbf{R}$ be a given measureable function which is bounded on every compact subset of U. Then, if ρ-weighted volume is defined for a subset $X \subset U$, ρ-weighted volume is also defined for $F(X)$, ρ_F-weighted volume is defined for X, where*

$$\rho_F(\bar{x}) = \begin{cases} \rho(F(x))|\det(dF/dx(\bar{x}))|, & dF/dx \text{ defined for } \bar{x}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$V_\rho(F(X)) = V_{\rho_F}(X).$$

Note that the formula is not always valid for non-injective functions (because of possible "folding"). It is also useful to remember that image of a line segment (zero Lebesque volume when $n > 1$) under a continuous map could cover a cube (positive Lebesque volume).

11.1.3 Volume change under a differential flow

Let consider the case when the map $F = S_t$ is defined by a smooth differential flow. Remember that, for a differential function $g : \mathbf{R}^n \mapsto \mathbf{R}^n$, $\mathrm{div}(g)$ is the trace of the Jacobian of g.

Theorem 11.2 *Let U be an open subset of $\mathbf{R}^n$. Let $f : U \mapsto \mathbf{R}^n$ and $\rho : U \mapsto \mathbf{R}$ be continuously differentiable functions. For $T > 0$ let U_T be the set of vectors $\bar{x} \in U$ such that the ODE*

$$\dot{x}(t) = f(x(t)),$$

has a solution $x : [0, T] \mapsto U$ such that $x(0) = \bar{x}$. Let $S_T : U_T \mapsto U$ be the map defined by $S_T(x(0)) = x(T)$. Then, if X is contained in a compact subset of U_T and has a ρ-weighted volume, the map $t \mapsto V_\rho(S_t(X))$ is well defined, differentiable, and its derivative at $t = 0$ is given by

$$\frac{dV_\rho(S_t(X))}{dt} = V_{\mathrm{div}(\rho f)}(X).$$

Proof According to Theorem 11.2,

$$V_\rho(S_t(X)) = \int_X \rho(S_t(\bar{x}))|\det(dS_t(x)/dx(\bar{x}))|d\bar{x}.$$

Note that

$$\left.\frac{dS_t(\bar{x})}{dt}\right|_{t=0} = f(\bar{x}),$$

and $dS_t(x)/dx(\bar{x}) = \Delta(t, \bar{x})$, where

$$\frac{d\Delta(t, \bar{x})}{dt} = \left.\frac{df}{dx}\right|_{x=S_t(\bar{x})} \Delta(t, \bar{x}), \quad \Delta(0, \bar{x}) = I.$$

Hence $\det(dS_t(x)/dx) > 0$, and, at $t = 0$,

$$\begin{aligned} \frac{d}{dt}|\det(dS_t(x)/dx(\bar{x}))| &= \frac{d}{dt}\det(\Delta(t, \bar{x})) \\ &= \mathrm{trace}\left.\frac{d\Delta(t, \bar{x})}{dt}\right|_{t=0} \\ &= \mathrm{div}(f)(\bar{x}), \end{aligned}$$

where the equality

$$\frac{d}{d\tau}\det(A(\tau)) = \det(A(\tau))\mathrm{trace}\left(A(\tau)^{-1}\frac{dA(\tau)}{d\tau}\right)$$

was used. Finally, at $t = 0$,

$$\frac{d}{dt}\rho(S_t(\bar{x}))|\det(dS_t(x)/dx(\bar{x}))| = (\nabla\rho)(\bar{x})f(\bar{x}) + \rho(\bar{x})\mathrm{div}(f)(\bar{x}) = \mathrm{div}(\rho f)(x).$$

■

11.2 Using volume monotonicity in system analysis

Results from the previous section allow one to establish invariance (monotonicity) of weighted volumes of sets evolving according to dynamical system equations. This section discusses application of such invariance in stability analysis.

11.2.1 Volume monotonicity and stability

Given an ODE model

$$\dot{x}(t) = f(x(t)), \tag{11.1}$$

where $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a continuously differentiable function, condition $\mathrm{div}(f) < 0$, if satisfied everywhere except possibly a set of zero volume, guarantees strictly monotonic decrease of Lebesque volume of sets of positive volume. This, however, does not guarantee stability. For example, the ODE

$$\begin{aligned} \dot{x}_1 &= -2x_1, \\ \dot{x}_2 &= x_2 \end{aligned}$$

does not have a stable equilibrium, while volumes of sets are strictly decreasing with its flow.

However, it is possible to make an opposite statement that a system for which a positively weighted volume is strictly monotonically increasing cannot have a stable equilibrium.

Theorem 11.3 *Let $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ and $\rho : \mathbf{R}^n \mapsto \mathbf{R}$ be continuously differentiable functions such that ρ-weighted volume of every ball in $\mathbf{R}^n$ is positive, and* $\mathrm{div}(f\rho)(\bar{x}) \geq 0$ *for all $\bar{x} \in \mathbf{R}^n$. Then system (11.1) has no asymptotically stable equilibria and no asymptotically stable limit cycles.*

Proof Assume to the contrary that $x_0 : \mathbf{R} \mapsto \mathbf{R}^n$ is a stable equilibrium or a stable limit cycle solution of (11.1). Then there exists $\epsilon > 0$ such that

$$\lim_{t\to\infty} \min_{\tau} |x(t) - x_0(\tau)| = 0$$

for every solution $x = x(\cdot)$ of (11.1) such that $x(0)$ belongs to the ball

$$B_0 = \{\bar{x} : \ |\bar{x} - x_0(0)| \leq \epsilon.$$

Let $v(t) = V_\rho(S_t(B_0))$, where S_t is the system flow. By assumption, v is monotonically non-increasing, $v(0) = 0$, and $v(t) \to 0$ as $t \to \infty$. The contradiction proves the theorem.
■

11.2.2 Volume monotonicity and strictly invariant sets

Let us call a set $X \subset \mathbf{R}^n$ *strictly invariant* for system (11.1) if every maximal solution $x = x(t)$ of (11.1) with $x(0) \in X$ is defined for all $t \in \mathbf{R}$ and stays in X for all $t \in \mathbf{R}$. Obviously, if X is a strictly invariant set then, for every weight ρ, $V_\rho(S_t(X))$ does not change as t changes. Therefore, if one can find a ρ for which $\operatorname{div}(\rho f) > 0$ almost everywhere, the strict invariance of X should imply that X is a set of a zero Lebesque volume, i.e. the following theorem is true.

Theorem 11.4 *Let U be an open subset of $\mathbf{R}^n$. Let $f : U \mapsto \mathbf{R}^n$ and $\rho : U \mapsto \mathbf{R}$ be continuously differentiable functions. Assume that* $\operatorname{div}(f\rho) > 0$ *for almost all points of U. Then, if X is a bounded closed subset of U which is* strictly invariant *for system (11.1), the Lebesque volume of X equals zero.*

As a special case, when $n = 2$ and $\rho \equiv 1$, we get the *Bendixon theorem*, which claims that if, in a simply connected region U, $\div(f) > 0$ almost everywhere, there exist no non-equilibrium periodic trajectories of (11.1) in U. Indeed, a non-equilibrium periodic trajectory on a plane bounds a strictly invariant set.

11.2.3 Monotonicity of singularly weighted volumes

So far, we considered weights which were bounded in the regions of interest. A recent observation by A. Rantzer shows that, when studying asymptotic stability of an equilibrium, it is most beneficial to consider weights which are singular at the equilibrium.

In particular, he has proven the following stability criterion.

Theorem 11.5 *Let $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ and $\rho : \mathbf{R}^n/\{0\} \mapsto \mathbf{R}$ be continuously differentiable functions such that $f(0) = 0$, $\rho(x)f(x)/|x|$ is integrable over the set $|x| \geq 1$, and* $\operatorname{div}(f\rho) > 0$ *for almost all $x \in \mathbf{R}^n$. If either $\rho \geq 0$ or 0 is a locally stable equilibrium of (11.1) then for almost all initial states $x(0)$ the corresponding solution $x = x(t)$ of (11.1) converges to zero as $t \to \infty$.*

To prove the statement for the case when $x = 0$ is a stable equilibrium, for every $r > 0$ consider the set X_r of initial conditions $x(0)$ for which

$$\sup_{t \in [T,\infty)} |x(t)| > r \quad \forall\, T > 0.$$

The set X is strictly invariant with respect to the flow of (11.1), and has well defined ρ-weighted volume. Hence, by the strict ρ-weighted volume monotonicity, the Lebesque measure of X_r equals zero. Since this is true for all $r > 0$, almost every solution of (11.1) converges to the origin.

Example 11.1 (Rantzer) The system

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_1^2 - x_2^2 \\ \dot{x}_2 &= -6x_2 + 2x_1x_2 \end{aligned}$$

satisfies conditions of Theorem 11.5 with $\rho(x) = |x|^{-4}$.