15.081 Fall 2009 Recitation 2

1 Linear Algebra Review

Read Section 1.5 of BT. Important concepts:

- linear independence of vectors
- subspace, basis, dimension
- the span of a collection of vectors
- the rank of a matrix
- nullspace, column space, row space

2 BT Exercise 2.10

Problem

Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$. Suppose that the matrix A has dimensions $m \times n$ and that its rows are linearly independent. For each one of the following statements, state whether it is true or false. If true, provide a proof, else, provide a counterexample.

- 1. If n = m + 1, then P has at most two basic feasible solutions.
- 2. The set of all optimal solutions is bounded.
- 3. At every optimal solution, no more than m variables can be positive.
- 4. If there is more than one optimal solution, then there are uncountably many optimal solutions.
- 5. If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.
- 6. Consider the problem of minimizing $\max\{c'x, d'x\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

Solution

- 1. **True**. The set P lies in an affine subspace defined by m = n 1 linearly independent constraints, that is, of dimension one. Hence, every solution of Ax = b is of the form $x^0 + \lambda x^1$, where x^0 is an element of P and x^1 is a nonzero vector. This, P is contained in a line and cannot have more than two extreme points. (If it had three, the one "in the middle" would be a convex combination of the other two, hence not an extreme point).
- 2. **False**. Consider minimizing 0, subject to $x \ge 0$. The optimal solution set $[0, \infty)$ is unbounded.
- 3. **False**. Consider a standard form problem with c = 0. Then, any feasible x is optimal, no matter how many positive components it has.
- 4. **True.** If x and y are optimal, so is any convex combination of them.
- 5. **False**. Consider the problem of minimizing x_2 subject to $(x_1, x_2) \ge (0, 0)$ and $x_2 = 0$. Then the set of all optimal solutions is the set $\{(x_1, 0)|x_1 \ge 0\}$. There are several optimal solutions, but only one optimal BFS.
- 6. **False**. Consider the problem of minimizing $|x_1-0.5| = \max\{x_1-0.5x_3, -x_1+0.5x_3\}$ subject to $x_1+x_2=1, x_3=1$ and $(x_1,x_2,x_3) \geq (0,0,0)$. Its unique optimal solution is $(x_1,x_2,x_3) = (0.5,0.5,1)$ which is not a BFS.

3 BFS

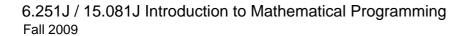
Consider a polyhedron P defined by linear equality and inequality constraints, and let $x^* \in \mathbb{R}^n$. Then

- 1. The vector x^* is a **basic solution** if:
 - All equality constraints are active;
 - Out of the constraints that are active at x^* , there are n of them that are linearly independent.
- 2. If x^* is a basic solution that satisfies all of the constraints, we say that it is a **basic feasible solution**.

4 Degeneracy

- 1. A basic solution $x \in \mathbb{R}^n$ of a linear optimization problem is said to be **degenerate** if there are more than n constraints which are active at x.
- 2. Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ and let x be a basic solution. Let m be the number of rows of A. The vector x is a **degenerate** basic solution if more than n-m of the components of x are zero.

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