

6.254 : Game Theory with Engineering Applications
Lecture 18: Games with Incomplete Information:
Bayesian Nash Equilibria and Perfect Bayesian Equilibria

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Outline

- Bayesian Nash Equilibria.
- Auctions.
- Extensive form games of incomplete information.
- Perfect Bayesian (Nash) Equilibria.

Incomplete Information

- In the last lecture, we studied incomplete information games where one agent is unsure about the payoffs or preferences of others.
- Examples abundant:
 - Bargaining, auctions, market competition, signaling games, social learning.
- We modeled such games as **Bayesian games** that consist of
 - A set of players \mathcal{I} ;
 - A set of actions (pure strategies) for each player i : S_i ;
 - A set of **types** for each player i : $\theta_i \in \Theta_i$;
 - A payoff function for each player i : $u_i(s_1, \dots, s_I, \theta_1, \dots, \theta_I)$;
 - A (joint) probability distribution $p(\theta_1, \dots, \theta_I)$ over types (or $P(\theta_1, \dots, \theta_I)$ when types are not finite).

Bayesian Games

- Importantly, throughout in Bayesian games, the strategy spaces, the payoff functions, possible types, and the prior probability distribution are assumed to be **common knowledge**.

Definition

A (pure) strategy for player i is a map $s_i : \Theta_i \rightarrow S_i$ prescribing an action for each possible type of player i .

- Given $p(\theta_1, \dots, \theta_I)$, player i can compute the conditional distribution $p(\theta_{-i} \mid \theta_i)$ using **Bayes rule**, where $\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_I)$.
- Player i knows her own type and evaluates her expected payoffs according to the conditional distribution $p(\theta_{-i} \mid \theta_i)$.

Bayesian Nash Equilibria

Definition (Bayesian Nash Equilibrium)

The strategy profile $s(\cdot)$ is a (pure strategy) Bayesian Nash equilibrium if for all $i \in \mathcal{I}$ and for all $\theta_i \in \Theta_i$, we have that

$$s_i(\theta_i) \in \arg \max_{s'_i \in S_i} \sum_{\theta_{-i}} p(\theta_{-i} | \theta_i) u_i(s'_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}),$$

or in the non-finite case,

$$s_i(\theta_i) \in \arg \max_{s'_i \in S_i} \int u_i(s'_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) P(d\theta_{-i} | \theta_i) .$$

- Hence a Bayesian Nash equilibrium is a Nash equilibrium of the “expanded game” in which each player i 's space of pure strategies is the set of maps from Θ_i to S_i .

Auctions

- A major application of Bayesian games is to **auctions**.
- This corresponds to a situation of incomplete information because the valuations of different potential buyers are unknown.
- We made the distinction between:
 - **Private value auctions**: valuation of each agent is independent of others' valuations;
 - **Common value auctions**: the object has a potentially common value, and each individual's signal is imperfectly correlated with this common value.
- We have analyzed private value first-price and second-price sealed bid auctions.
 - Each of these two auction formats defines a static game of incomplete information (Bayesian game) among the bidders.
 - We determined Bayesian Nash equilibria in these games and compared the equilibrium bidding behavior.

Model

- There is a single object for sale and N potential buyers bidding for it.
- Bidder i assigns a value v_i to the object, i.e., a utility $v_i - b_i$, when he pays b_i for the object. He knows v_i . This implies that we have a **private value** auction (v_i is his “private information” and “private value”).
- Suppose also that each v_i is **independently and identically distributed** on the interval $[0, \bar{v}]$ with cdf F , with continuous density f and full support on $[0, \bar{v}]$.
- Bidder i knows the realization of its value v_i and that other bidders' values are independently distributed according to F , i.e., all components of the model except the realized values are “common knowledge”.
- Bidders are *risk neutral*, i.e., they are interested in maximizing their expected profits.
- This model defines a Bayesian game of incomplete information, where the types of the players (bidders) are their valuations, and a pure strategy for a bidder is a map

$$\beta_i : [0, \bar{v}] \rightarrow \mathbb{R}_+.$$

Results

- With a reasoning similar to its counterpart with complete information, we establish in a second-price auction, it is a weakly dominant strategy to bid *truthfully*, i.e., according to $\beta^{\text{II}}(v) = v$.

Proposition

In the second price auction, there exists a unique Bayesian Nash equilibrium which involves

$$\beta^{\text{II}}(v) = v.$$

- For first-price auctions, we looked for a **symmetric (increasing and differentiable) equilibrium**.

Proposition

In the first price auction, there exists a symmetric equilibrium given by

$$\beta^{\text{I}}(v) = \mathbb{E}[y_1 \mid y_1 < v].$$

Results

- We also showed that both auction formats yield the same expected revenue to the seller.
- Moreover, we established the **revenue-equivalence theorem**:

Theorem

Any symmetric and increasing equilibria of any standard auction (such that the expected payment of a bidder with value 0 is 0) yields the same expected revenue to the seller.

Common Value Auctions: A Simple Example

- Common value auctions are more complicated, because each player has to infer the valuation of the other player (which is relevant for his own valuation) from the bid of the other player (or more generally from the fact that he has one).
- The analysis of common value auctions is typically more complicated. So we will just communicate the main ideas using an example.
- Consider the following example. There are two players, each receiving a signal s_i . The value of the good to both of them is

$$v_i = \alpha s_i + \gamma s_{-i},$$

where $\alpha \geq \gamma \geq 0$. Private values are the special case where $\alpha = 1$ and $\gamma = 0$.

- Suppose that both s_1 and s_2 are distributed uniformly over $[0, 1]$.

Second Price Auctions with Common Values

- Now consider a second price auction.
- Instead of truthful bidding, now the symmetric equilibrium is each player bidding

$$\beta_i(s_i) = (\alpha + \gamma) s_i.$$

- Why?
- Given that the other player is using the same strategy, the probability that player i will win when he bids b is

$$\begin{aligned}\Pr(\beta_{-i}(s_{-i}) < b) &= \Pr((\alpha + \gamma) s_{-i} < b) \\ &= \frac{b}{\alpha + \gamma}.\end{aligned}$$

- The price he will pay is simply $\beta_{-i}(s_{-i}) = (\alpha + \gamma) s_{-i}$ (since this is a second price auction).

Second Price Auctions with Common Values (continued)

- Conditional on the fact that $b_{-i} \leq b$ (i.e., winning), we can compute the expected payment as

$$\mathbb{E} \left[(\alpha + \gamma) s_{-i} \mid s_{-i} < \frac{b}{\alpha + \gamma} \right] = \frac{b}{2}.$$

- Next, let us compute the expected value of player $-i$'s signal conditional on player i winning. With the same reasoning, this is

$$\mathbb{E} \left[s_{-i} \mid s_{-i} < \frac{b}{\alpha + \gamma} \right] = \frac{b}{2(\alpha + \gamma)}.$$

Second Price Auctions with Common Values (continued)

- Therefore, the expected utility of bidding b_i for player i with signal s_i is:

$$\begin{aligned} U_i(b_i, s_i) &= \Pr[b_i \text{ wins}] \times \left(\alpha s_i + \gamma \mathbb{E}[s_{-i} \mid b_i \text{ wins}] - \frac{b_i}{2} \right) \\ &= \frac{b_i}{\alpha + \gamma} \left[\alpha s_i + \frac{\gamma}{2} \frac{b_i}{\alpha + \gamma} - \frac{b_i}{2} \right]. \end{aligned}$$

- Maximizing this with respect to b_i (for given s_i) implies

$$\beta_i(s_i) = (\alpha + \gamma) s_i,$$

establishing that this is a symmetric Bayesian Nash equilibrium of this common value auction.

First Price Auctions with Common Values

- We can also analyze the same game under an auction format corresponding to first price sealed bid auctions.
- In this case, with an analysis similar to that of the first price auctions with private values, we can establish that the unique symmetric Bayesian Nash equilibrium is for each player to bid

$$\beta_i^I(s_i) = \frac{1}{2} (\alpha + \beta) s_i.$$

- It can be verified that expected revenues are again the same. This illustrates the general result that revenue equivalence principle continues to hold for common value auctions.

Incomplete Information in Extensive Form Games

- Many situations of incomplete information cannot be represented as static or strategic form games.
- Instead, we need to consider extensive form games with an explicit order of moves—or dynamic games.
- In this case, as mentioned earlier in the lectures, we use **information sets** to represent what each player knows at each stage of the game.
- Since these are dynamic games, we will also need to strengthen our Bayesian Nash equilibria to include the notion of **perfection**—as in subgame perfection.
- The relevant notion of equilibrium will be **Perfect Bayesian Equilibria**, or Perfect Bayesian Nash Equilibria.

Example

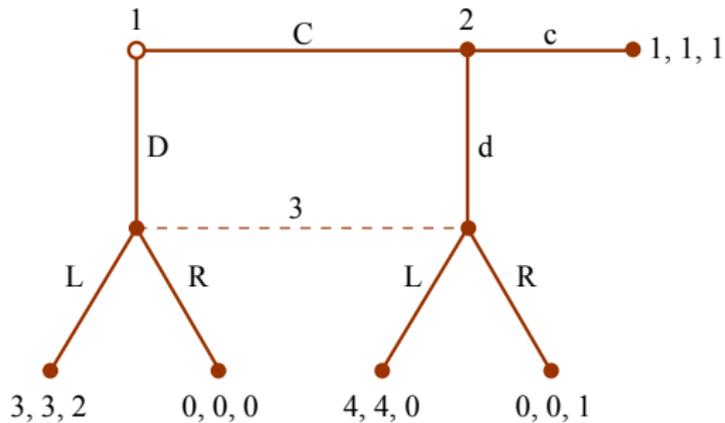


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Figure: Selten's Horse

Dynamic Games of Incomplete Information

Definition

A dynamic game of incomplete information consists of

- *A set of players \mathcal{I} ;*
- *A sequence of histories H^t at the t^{th} stage of the game, each history assigned to one of the players (or to Nature/Chance);*
- *An information partition, which determines which of the histories assigned to a player are in the same information set.*
- *A set of (pure) strategies for each player i , S_i , which includes an action at each information set assigned to the player.*
- *A set of types for each player i : $\theta_i \in \Theta_i$;*
- *A payoff function for each player i : $u_i(s_1, \dots, s_I, \theta_1, \dots, \theta_I)$;*
- *A (joint) probability distribution $p(\theta_1, \dots, \theta_I)$ over types (or $P(\theta_1, \dots, \theta_I)$ when types are not finite).*

Strategies, Beliefs and Bayes Rule

- The most economical way of approaching these games is to first define a **belief system**, which determines a posterior for each agent over the set of nodes in an information set. Belief systems are often denoted by μ .
- In Selten's horse player 3 needs to have beliefs about whether when his information set is reached, he is at the left or the right node.
- A **behavioral strategy** of player i in an extensive form game is a function that assigns to each of i 's information sets a probability distribution over the possible actions at that information set (each of these distributions are independent).
- We say that a strategy is **sequentially rational** if, given beliefs and other players strategies, no player can improve his or her payoffs at any stage of the game.
- We say that a belief system is **consistent** if it is derived from equilibrium strategies using Bayes rule.

Strategies, Beliefs and Bayes Rule (continued)

- In Selten's horse, if the strategy of player 1 is D , then Bayes rule implies that $\mu_3(\text{left}) = 1$, since conditional on her information set being reached, player 3's assessment must be that this was because player 1 played D .
- Similarly, if the strategy of player 1 is D with probability p and the strategy of player 2 is d with probability q , then Bayes rule implies that

$$\mu_3(\text{left}) = \frac{p}{p + (1 - p)q}.$$

- What happens if $p = q = 0$? In this case, $\mu_3(\text{left})$ is given by $0/0$, and is thus *undefined*. Under the consistency requirement here, it can take any value. This implies, in particular, that information sets that are not reached along the equilibrium path will have **unrestricted** beliefs.

Perfect Bayesian Equilibria

Definition

A *Perfect Bayesian Equilibrium* in a dynamic game of incomplete information is a strategy profile s and belief system μ such that:

- *The strategy profile s is sequentially rational given μ* (each player's strategy is optimal in the part of the game that follows each of her information sets, given the strategy profile and her belief about the history in the information set that has occurred).
- *The belief system μ is consistent given s* (for every information set reached with positive probability given the strategy profile s , the probability assigned to each history in the information set by the belief system μ is given by Bayes' rule).
- Perfect Bayesian Equilibrium is a relatively weak equilibrium concept for dynamic games of incomplete information. It is often strengthened by restricting beliefs of information sets that are not reached along the equilibrium path.

Perfect Bayesian Equilibria

Theorem

Consider a finite dynamic game of incomplete information. Then a (possibly mixed) Perfect Bayesian Equilibrium exists.

- Once again, the idea of the proof is the same as those we have seen before.
- Backward induction starting from the information sets at the end ensures perfection, and one can construct a belief system supporting these strategies, so the result is a Perfect Bayesian Equilibrium.

Theorem

The strategy profile in any Perfect Bayesian Equilibrium is a Nash Equilibrium.

- Sequential rationality implies each player's strategy optimal at the beginning of the game given others' strategies and beliefs. Consistency ensures correctness of the beliefs.

Perfect Bayesian Equilibria in Selten's Horse

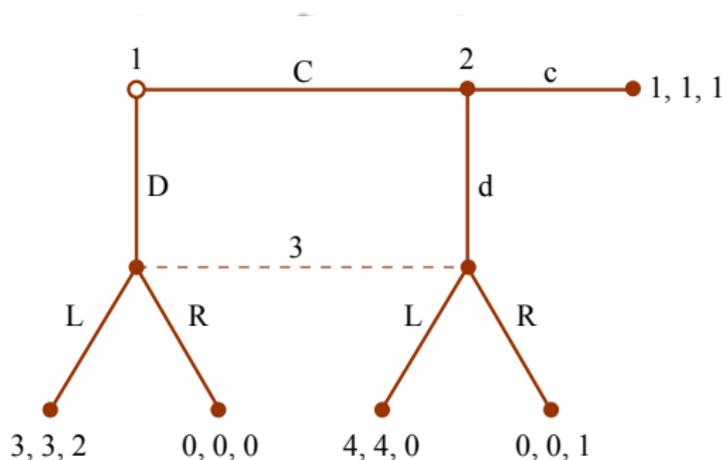


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- It can be verified that there are two pure strategy Nash equilibria.
 - (C, c, R) and (D, c, L) .

Perfect Bayesian Equilibria in Selten's Horse (continued)

- However, if we look at *sequential rationality*, the second of these equilibria will be ruled out.
- Suppose we have (D, c, L) .
- The belief of player 3 will be $\mu_3(\text{left}) = 1$.
- Player 2, if he gets a chance to play, will then never play c , since d has a payoff of 4, while c would give him 1. If he were to play d , then player 1 would prefer C , but (C, d, L) is not an equilibrium, because then we would have $\mu_3(\text{left}) = 0$ and player 3 would prefer R .
- Therefore, there is a unique pure strategy Perfect Bayesian Equilibrium outcome (C, c, R) . The belief system that supports this could be any $\mu_3(\text{left}) \in [0, 1/3]$.

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