6.262: Discrete Stochastic Processes 3/9/11

Lecture 11 : Renewals: strong law and rewards

Outline:

- **Review of convergence WP1**
- **Review of SLLN**
- **The strong law for renewal processes**
- **Central limit theorem (CLT) for Renewals**
- **Time average residual life**

Theorem 1: let $\{Y_n;\, n \geq 1\}$ **be rv's satisfying** $\sum_{n=1}^{\infty} \mathsf{E}[|Y_n|] < \infty$**. Then** $\Pr\{\omega : \lim_{n\to\infty} Y_n(\omega) = 0\} = 1$. **(conv. WP1)**

Note 1:

$$\lim_{m\to\infty} \sum_{n=1}^{m} \mathsf{E}[|Y_n|] = \sum_{n=1}^{\infty} \mathsf{E}[|Y_n|] < \infty$$

means that the limit exists, and thus that

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} \mathsf{E}[|Y_n|] = 0 \qquad (1)$$

This is a stronger requirement than $\lim_n \mathsf{E}[|Y_n|] = 0$**.**

The requirement $\lim_n \mathsf{E}[|Y_n|] = 0$ **implies that** $\{Y_n;\, n \geq 1\}$ **converges to 0 in probability (see problem set).**

The stronger requirement in (1) lets us say something about entire sample paths.

Review of SLLN

Note 2: The usefulness of Theorem 1 depends on how the rv's $Y_1, Y_2, \ldots$ are chosen.

For SLLN (assuming $\overline{X} = 0$ and $\mathsf{E}\left[X^4\right] < \infty$), we chose $Y_n = S_n^4/n^4$ where $S_n = X_1 + \cdots + X_n$.

Since $\mathsf{E}\left[S_n^4\right] \sim n^2$, we saw that $\sum_n \mathsf{E}\left[S_n^4\right]/n^4 < \infty$.

Thus $\Pr\left\{\lim_{n\to\infty} S_n^4/n^4 = 0\right\} = 1$.

If $S_n^4(\omega)/n^4 = a_n$, then $|S_n(\omega)|/n = a_n^{1/4}$. If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n^{1/4} = 0$. Thus

$$\Pr\left\{\lim_{n\to\infty} S_n/n = 0\right\} = 1.$$

This is the SLLN for $\overline{X} = 0$ and $\mathsf{E}\left[X^4\right] < \infty$.

This all becomes slightly more general if $\{Z_n; n \geq 1\}$ is said to converge to a constant α WP1 if $\Pr\{\lim_{n\to\infty} Z_n(\omega) = \alpha\} = 1$.

Note that $\{Z_n; n \geq 1\}$ converges to α if and only if $\{Z_n - \alpha; n \geq 1\}$ converges to 0.

Similarly, if $\{X_n;\, n \geq 1\}$ is an IID sequence with $\overline{X} = \alpha$, then $\{X_n - \alpha;\, n \geq 1\}$ is IID with mean 0.

For renewal processes, the inter renewal intervals are positive, so it is more convenient to leave the mean in.

The strong law for renewal processes

A major factor in making the SLLN and convergence WP1 easy to work with was illustrated in going from S_n^4/n^4 to $|S_n|/n$ above.

The following theorem generalizes this.

Theorem 2: Assume that $\{Z_n;\, n \geq 1\}$ converges to α WP1 and assume that $f(x)$ is a real valued function of a real variable that is continuous at $x = \alpha$. Then $\{f(Z_n);\, n \geq 1\}$ converges WP1 to $f(\alpha)$.

Example 1: If $f(x) = x + \beta$ for some constant β, and Z_n converges to α, then $U_n = Z_n + \beta$ converges to $\alpha + \beta$, corresponding to a trivial change of mean.

Example 2: If $f(x) = x^{1/4}$ for $x \geq 0$, and $Z_n \geq 0$ converges to 0 WP1, then $f(Z_n)$ converges to $f(0)$.

Thm: $\lim_n Z_n = \alpha$ WP1 and $f(x)$ continuous at α implies $\lim_n f(Z_n) = f(\alpha)$ WP1.

Pf: For each ω such that $\lim_n Z_n(\omega) = \alpha$, we use the result for a sequence of numbers that says $\lim_n f(Z_n(\omega)) = f(\alpha)$.

For renewal processes, each inter-renewal interval X_i is positive and (assuming that $\mathsf{E}[X]$ exists), $\mathsf{E}[X] > 0$ (see Exercise 4.2a). Thus $\mathsf{E}[S_n/n] > 0$.

The SLLN then applies, and

$$\Pr\Big\{\omega : \lim_{n\to\infty} S_n(\omega)/n = \overline{X}\Big\} = 1.$$

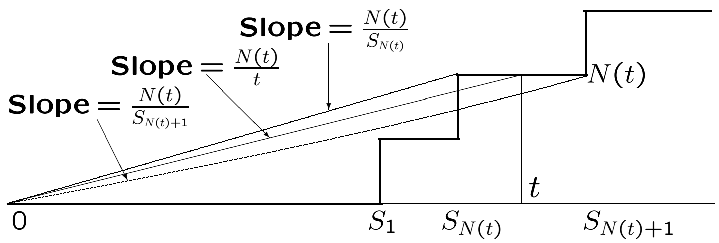
Using $f(x) = 1/x$,

$$\Pr\left\{\omega : \lim_{n\to\infty} \frac{n}{S_n(\omega)} = \frac{1}{\overline{X}}\right\} = 1.$$

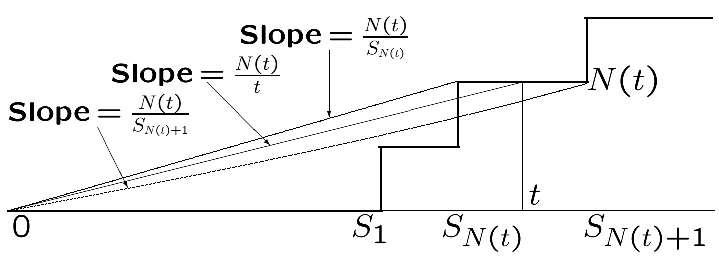
$$\Pr\left\{\omega : \lim_{n\to\infty} \frac{n}{S_n(\omega)} = \frac{1}{\overline{X}}\right\} = 1.$$

This implies the strong law for renewal processes:
Theorem 3: (strong law for renewal processes) For a renewal process with $\overline{X} < \infty$,

$$\Pr\left\{\omega : \lim_{t\to\infty} N(t)/t = 1/\overline{X}\right\} = 1.$$



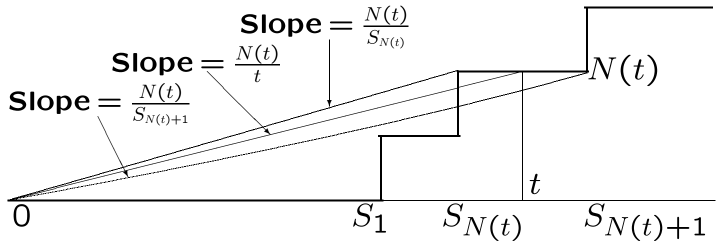
Note that $N(t)/t \le N(t)/S_{N(t)}$, and that $N(t)/S_{N(t)}$ goes through the same set of values as n/S_n



Note that $N(t)/t \le N(t)/S_{N(t)}$, and that $N(t)/S_{N(t)}$ goes through the same set of values as n/S_n, i.e.,

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)}} = \lim_{n\to\infty} \frac{n}{S_n} = \frac{1}{\overline{X}} \qquad \textbf{WP1}.$$

This assumes that $\lim_{t\to\infty} N(t) = \infty$ WP1, which is demonstrated in the text.

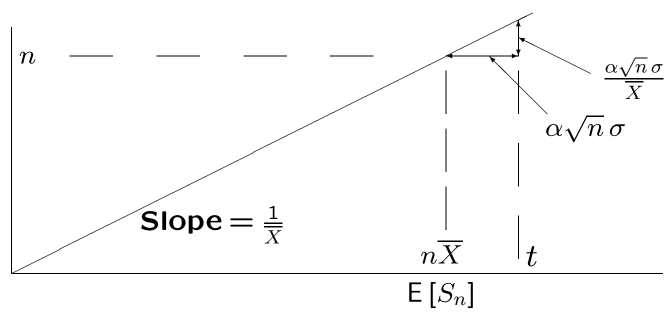


Note that $N(t)/t \ge N(t)/S_{N(t)+1}$, and that $N(t)/S_{N(t)+1}$ goes through the same set of values as $n/(S_{n+1})$, i.e.,

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)+1}} = \lim_{n\to\infty} \frac{n}{S_{n+1}}$$
$$= \lim_{n\to\infty} \frac{n+1}{S_{n+1}} \frac{n}{n+1} = \frac{1}{\overline{X}} \qquad \textbf{WP1.}$$

Since $N(t)/t$ is between these two quantities with the same limit, $\lim_n N(t)/t = 1/\overline{X}$ WP1.

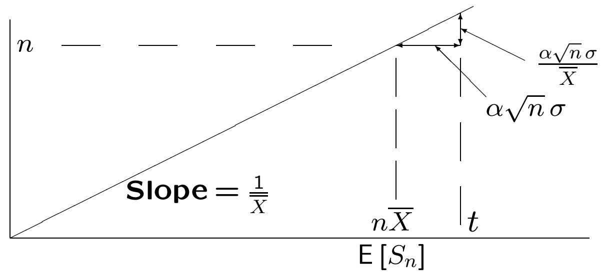
<u>Central limit theorem (CLT) for Renewals</u>



$$\Pr\{S_n \le t\} \approx \Phi(\alpha); \quad t = n\overline{X} + \alpha\sigma\sqrt{n}$$

$$\Pr\{N(t) \ge n\} \approx \Phi(\alpha); \quad n = \frac{t}{\overline{X}} - \frac{\alpha\sigma\sqrt{n}}{\overline{X}} \approx \frac{t}{\overline{X}} - \frac{\alpha\sigma\sqrt{t}}{\overline{X}^{3/2}}$$

$$\Pr\left\{N(t) \ge \frac{t}{\overline{X}} - \frac{\alpha\sigma\sqrt{t}}{\overline{X}^{3/2}}\right\} \approx \Phi(\alpha);$$



$$\Pr\left\{N(t) \geq \frac{t}{\overline{X}} - \frac{\alpha\sigma\sqrt{t}}{\overline{X}^{3/2}}\right\} \approx \Phi(\alpha);$$

$$\Pr\left\{\frac{N(t) - t/\overline{X}}{\sigma\sqrt{t}\ \overline{X}^{-3/2}} \geq -\alpha\right\} \approx \Phi(\alpha)$$

$$\Pr\left\{\frac{N(t) - t/\overline{X}}{\sigma\sqrt{t}\ \overline{X}^{-3/2}} \leq -\alpha\right\} \approx 1 - \Phi(\alpha) = \Phi(-\alpha)$$

This is the CLT for renewal processes. $N(t)$ tends to Gaussian with mean $t/\overline{X}$ and s.d. $\sigma\sqrt{t}\ \overline{X}^{-3/2}$.

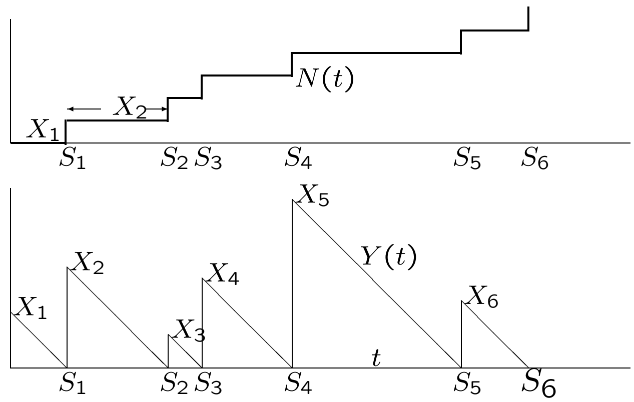
Time-average Residual life

Def: The residual life $Y(t)$ of a renewal process at time t is the remaining time until the next renewal, i.e., $Y(t) = S_{N(t)+1} - t$.

It's how long you have to wait for a bus (if bus arrivals were renewal processes).

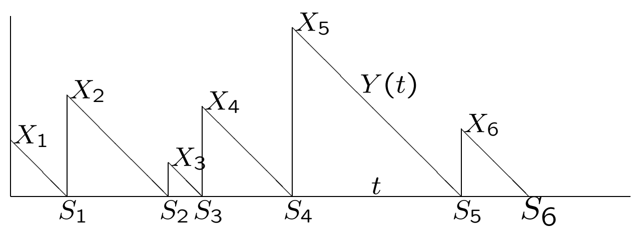
We can view residual life as a reward function on a renewal process. The sample reward at t is a function of the sample path of renewals.

The residual life, as a function of t, is a random process, and we can look at its time-average value, $\left[\int_0^t Y(\tau)\,d\tau\right]/t$.



Note that a residual-life sample function is a sequence of isosceles triangles, one starting at each arrival epoch. The time average for a given sample function is

$$\frac{1}{t}\int_0^t y(\tau)d\tau = \frac{1}{2t}\sum_{i=1}^{n(t)} x_i^2 + \frac{1}{t}\int_{\tau=s_{n(t)}}^t y(\tau)d\tau$$



$$\frac{1}{2t}\sum_{n=1}^{N(t)} X_n^2 \le \frac{1}{t}\int_0^t Y(\tau)d\tau \le \frac{1}{2t}\sum_{n=1}^{N(t)+1} X_n^2$$

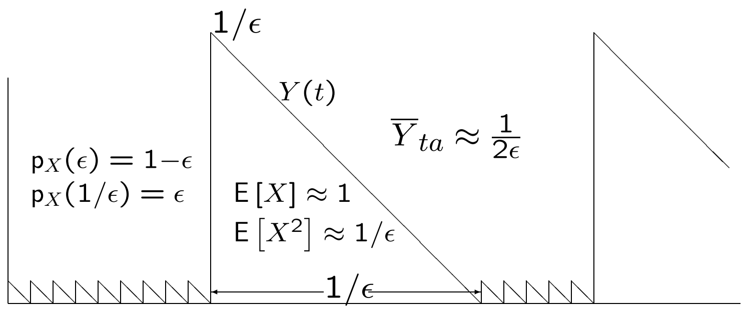
$$\lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} X_n^2}{2t} = \lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} X_n^2}{N(t)} \, \frac{N(t)}{2t} = \frac{\mathsf{E}\left[X^2\right]}{2\mathsf{E}\left[X\right]} \qquad \textbf{WP1}$$

$$\lim_{t\to\infty} \frac{\int_0^t Y(\tau)\,d\tau}{t} = \frac{\mathsf{E}\left[X^2\right]}{2\mathsf{E}\left[X\right]} \qquad \textbf{W.P.1}$$

Time average residual life $\overline{Y}_{ta} = \frac{\mathsf{E}[X^2]}{2\mathsf{E}[X]}$**.**

If X **is almost deterministic,** $\overline{Y}_{ta} \approx \mathsf{E}[X]/2$**.**

If X **exponential,** $\overline{Y}_{ta} = \mathsf{E}[X]$**.**



The expected duration between long intervals is $\approx$ **1.**

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