

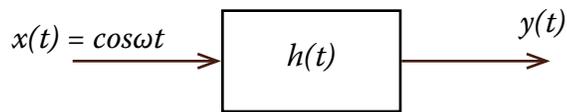
# 6.302 Feedback Systems

Recitation 4: Complex Variables and the s-Plane

Prof. Joel L. Dawson

Number 1 question: Why deal with imaginary and complex numbers at all?

One answer is that, as an analytical technique, they make our lives easier. Consider passing a cosine through an LTI filter with impulse response  $h(t)$ :



Because this is a linear and time-invariant system, we can apply the convolution integral to figure out  $y(t)$ :

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\cos\omega(t-\tau)d\tau$$

Evaluating this can be easy or hard, depending on the form of  $h(t)$ . But, if we take advantage of Euler's Identity,  $\cos\omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$  and so

$$\begin{aligned} y(t) &= \frac{1}{2} \int_{-\infty}^{\infty} h(\tau)[e^{j\omega(t-\tau)} + e^{-j\omega(t-\tau)}]d\tau \\ &= \frac{1}{2} \underbrace{\left[ \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau} d\tau \right]}_{H(j\omega)} e^{j\omega t} + \frac{1}{2} \underbrace{\left[ \int_{-\infty}^{\infty} h(\tau)e^{j(-\omega)\tau} d\tau \right]}_{H(-j\omega)} e^{-j\omega t} \end{aligned}$$

LAPLACE  
TRANSFORM

$$= \frac{1}{2} H(j\omega)e^{j\omega t} + \frac{1}{2} H(-j\omega)e^{-j\omega t}$$

Exponentials, even complex exponentials, are eigenfunctions of LTI systems.

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So complex numbers appear to be convenient. But in our systems, we're always dealing with "real" signals in the sense that there is no "other" or "imaginary" part. What is it about our LTI mathematical machinery that lets us plunge into the complex s-plane and re-emerge safely in the world of real signals?

Let's look at how we get into the s-plane in the first place. Starting with a real time-domain signal  $h(t)$ , we use the Laplace Transform evaluated on the  $j\omega$  axis:

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(t)e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} h(t)\cos\omega t dt - j \int_{-\infty}^{\infty} h(t)\sin\omega t dt \end{aligned}$$

We can say that  $H(j\omega)$  is conjugate symmetric:

$$\begin{aligned} \operatorname{Re} \{H(j\omega)\} &= \operatorname{Re} \{H(-j\omega)\} \\ \operatorname{Im} \{H(j\omega)\} &= -\operatorname{Im} \{H(-j\omega)\} \end{aligned}$$

So the real part of  $H(j\omega)$  is an even function of  $\omega$ , while the imaginary part is an odd function of  $\omega$ . Just the fact that  $h(t)$  is real guaranteed this.

Great. We're in the world of the s-plane now, and all of our system functions, frequency responses, etc., have this property of being conjugate symmetric. Convince yourself that any product of  $H(j\omega)G(j\omega)$  will be conjugate symmetric if  $H(j\omega)$  and  $G(j\omega)$  have this property separately.

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Time passes. We do our manipulations in the s-plane, passing through filters, amplifiers, and motors to our hearts' content. We have our output  $Y(j\omega)$ , and it's time to go back to the time domain. We pull out our trusty inverse Laplace Transform, and choose as our contour of integration the  $j\omega$  axis:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega) e^{j\omega t} d\omega$$

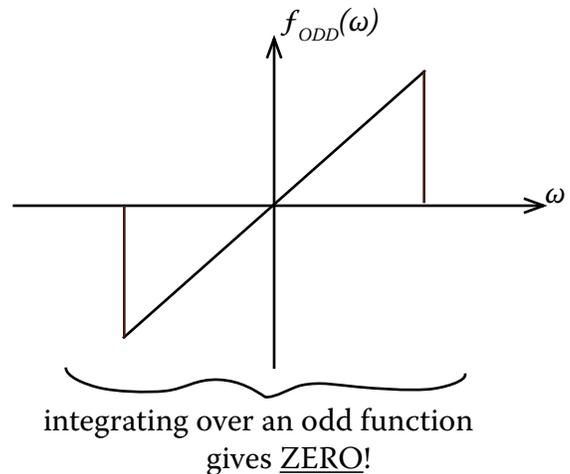
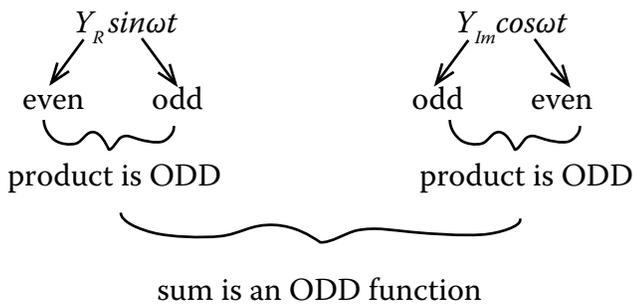
Let  $Y_R(j\omega) = \text{Re}\{Y(j\omega)\}$

$$Y_{Im}(j\omega) = \text{Im}\{Y(j\omega)\}$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (Y_R + jY_{Im})(\cos\omega t + j\sin\omega t) d\omega$$

$$= \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} (Y_R \cos\omega t - Y_{Im} \sin\omega t) d\omega}_{\text{real part}} + \underbrace{\frac{j}{2\pi} \int_{-\infty}^{\infty} (Y_R \sin\omega t - Y_{Im} \cos\omega t) d\omega}_{\text{imaginary part}}$$

Our hope is that the imaginary part is zero. Let's examine the terms:



We're left with the purely real function of time:  $y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (Y_R \cos\omega t - Y_{Im} \sin\omega t) d\omega$

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Our LTI machinery, then, gives us the convenience of complex variables and the guarantee that our results will be real functions in time.

## Properties of the Laplace Transform

DIFFERENTIATION: 
$$\mathcal{L} \left[ \frac{df}{dt} \right] = sF(s) - f(0)$$

INTEGRATION: 
$$\mathcal{L} [ \int_0^t f(\tau) d\tau ] = \frac{F(s)}{s}$$

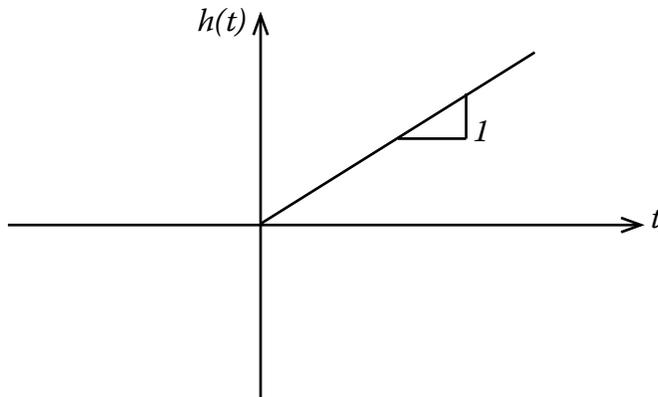
INITIAL VALUE THEOREM: 
$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

FINAL VALUE THEOREM: 
$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

We can use these properties to make our calculations easier.

Example 1: 
$$H(s) = \frac{1}{s^2} \quad ; \quad \text{sketch } h(t)$$

We know that  $\mathcal{L}^{-1} \{1/s\}$  is  $u(t)$ . Using the integration property, we know that  $h(t)$  must be  $\int_0^t u(t) dt$ , or a ramp:



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Example 2:

$$H(s) = \frac{s+2}{s+1} \quad \text{sketch the step response.}$$

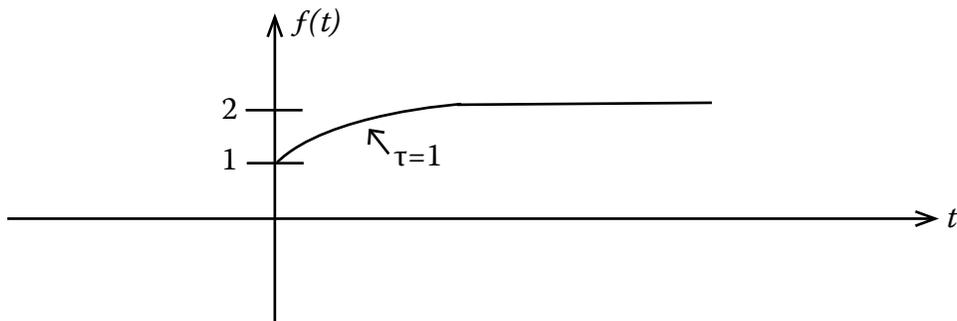
$$\text{Initial value: } \lim_{s \rightarrow \infty} s \left[ \frac{1}{s} \frac{s+2}{s+1} \right] = 1$$

$$\text{Final value: } \lim_{s \rightarrow 0} s \left[ \frac{1}{s} \frac{s+2}{s+1} \right] = 2$$

How it gets from start to finish:

$$\frac{1}{s} \frac{s+2}{s+1} = \frac{A}{s} + \frac{B}{s+1}$$

A step  $u(t)$                        $\sim e^{-t} u(t)$



Class Exercise:

$$\text{Sketch step response of } H(s) = \frac{1-s}{s+1}$$

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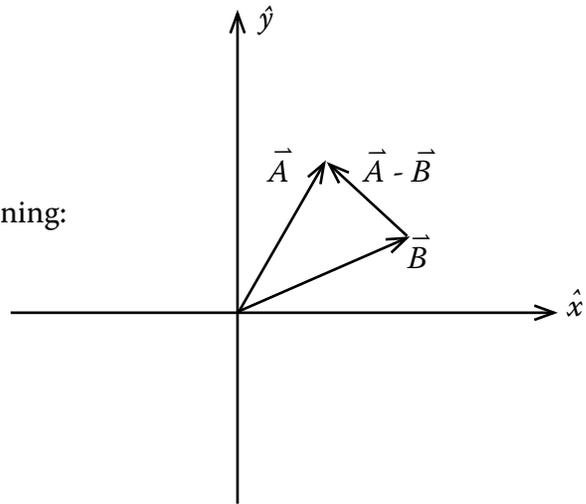
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Complex numbers and vector geometry

Recall that for ordinary vectors,  $\vec{A} = a_1 \hat{x} + a_2 \hat{y}$ .

And  $\vec{B} = b_1 \hat{x} + b_2 \hat{y}$ , the vector  $\vec{A} - \vec{B}$  had a very definite meaning:

$$\vec{A} - \vec{B} = (a_1 - b_1) \hat{x} + (a_2 - b_2) \hat{y}$$



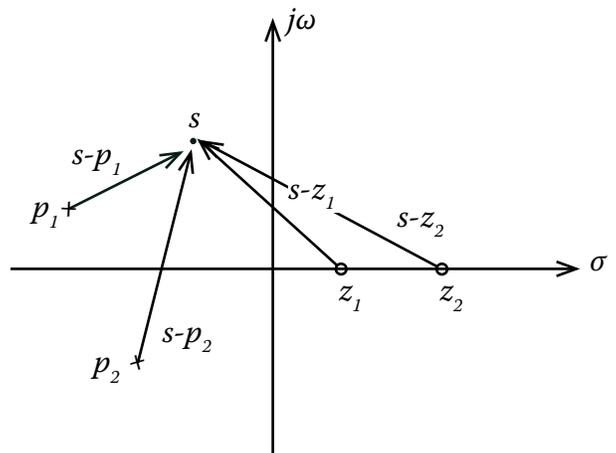
For complex numbers, we do a similar thing:

$$\left. \begin{aligned} u &= a_1 + ja_2 \\ v &= b_1 + jb_2 \end{aligned} \right\} u - v = (a_1 - b_1) + j(a_2 - b_2)$$

We treat complex numbers like vectors.

So with poles and zeros, we appeal to a geometrical picture.

$$F(s) = \frac{\text{vector } (s-z_1) \text{ vector } (s-z_2)}{\text{vector } (s-p_1) \text{ vector } (s-p_2)}$$



The vector  $s - z_1$  has a magnitude ( $R_{z1}$ ) and an angle, or phase  $\phi_{z1}$ . We can write  $F(s)$  as

$$F(s) = \frac{r_{z1} e^{j\phi_{z1}} r_{z2} e^{j\phi_{z2}}}{r_{p1} e^{j\phi_{p1}} r_{p2} e^{j\phi_{p2}}} = \frac{r_{z1} r_{z2}}{r_{p1} r_{p2}} e^{j(\phi_{z1} + \phi_{z2} - \phi_{p1} - \phi_{p2})}$$