MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2018
Problem Set 8

Readings:
Notes from Lecture 14 and 15.
[GS]: Section 4.9, 4.10, 5.7-5.9

Exercise 1. Let $\phi_A(t) = \mathbb{E}[e^{itA}]$ be a characteristic function of r.v. A.

(a) Find $\phi_X(t)$ if X is a Bernoulli(p) random variable.

(b) Suppose that $\phi_{X_i} = \cos(t/2^i)$. What is the distribution of X_i?

(c) Let $X_1, X_2, \ldots$ be independent and let $S_n = X_1 + \cdots + X_n$. Suppose that S_n converges almost surely to some random variable S. Show that $\phi_S(t) = \prod_{i=1}^{\infty} \phi_{X_i}(t)$.

(d) Evaluate the infinite product $\prod_{i=1}^{\infty} \cos(t/2^i)$. *Hint:* Think probabilistically; the answer is a very simple expression.

Exercise 2. Let X be a random variable with mean, variance, and moment generating function), denoted by $\mathbb{E}[X]$, $\text{var}(X)$, and $M_X(s)$, respectively. Similarly, let Y be a random variable associated with $\mathbb{E}[Y]$, $\text{var}(Y)$, and $M_Y(s)$. Each part of this problem introduces a new random variable Q, H, G, D. Determine the means and variances of the new random variables, in terms of the means, and variances of X and Y.

(a) $M_Q(s) = [M_X(s)]^5$.

(b) $M_H(s) = [M_X(s)]^3[M_Y(s)]^2$.

(c) $M_G(s) = e^{6s} M_X(s)$.

(d) $M_D(s) = M_X(6s)$.

Exercise 3. A random (nonnegative integer) number of people K, enter a restaurant with n tables. Each person is equally likely to sit on any one of the tables, independently of where the others are sitting. Give a formula, in terms of the moment generating function $M_K(\cdot)$, for the expected number of occupied tables (i.e., tables with at least one customer).

Exercise 4. (Problem 7, Section 4.9, [GS]): Let the vector $X_r, 1 \le r \le n$ have a multivariate normal distribution with zero means and covariance matrix $V = (v_{ij})$. Show that, conditional on the event $\sum_{i=1}^n X_r = x$, $X_1 \stackrel{d}{=} N(a,b)$, where $a = (\rho s/t)x, b = s^2(1-\rho^2)$ and $s^2 = v_{11}, t^2 = \sum_{ij} v_{ij}, \rho = \sum_i v_{i1}/(st)$.

Exercise 5. Suppose that for every k, the pair (X_k, Y) has a bivariate normal distribution. Furthermore, suppose that the sequence X_k converges to X, almost surely. Show that (X, Y) has a bivariate normal distribution. ***Hint:*** First show that if X_k is a sequence of normally distributed random variables which converges to X almost surely, then X has to be normally distributed as well. Then use the "right" definition of the bivariate normal.

Exercise 6. Suppose that $X, Z_1, \ldots, Z_n$ have a multivariate normal distribution, and X has zero mean. Furthermore, suppose that $Z_1, \ldots, Z_n$ are independent. Show that $\mathbb{E}[X \mid Z_1, \ldots, Z_n] = \sum_{i=1}^n \mathbb{E}[X \mid Z_i]$. Is this result true without the multivariate normal example? (Prove or give a counterexample.)

Exercise 7. Let $Y_1, \ldots, Y_n$ be independent N(0,1) random variables, and let $X_j = \sum_{r=1}^n c_{jr} Y_r$, for some constants c_{jr}. Show that

$$\mathbb{E}[X_j \mid X_k] = \left(\frac{\sum_r c_{jr} c_{kr}}{\sum_r c_{kr}^2} \right) X_k.$$

Exercise 8. [Optional, not for grade] Let X, Y be i.i.d. with finite second moments. Suppose that $X + Y$ and $X - Y$ are independent. Show that they must be Gaussian. (*Hint:* Derive a second order differential equation on $\phi_X(t)$.)

Exercise 9. [Optional, not for grade] (Problem 20 in p. 142, Section 4.14 of [GS]): Suppose that X and Y are independent and identically distributed, and not necessarily continuous random variables. Show that $X + Y$ cannot be uniformly distributed on $[0, 1]$.