Complements are independent too

Problem 0.1. *Let* $\{A_i\}_{i\in T}$ *be a (possibly infinite, possibly uncountable) set of independent events. Prove that* $\{A_i^c\}_{i\in T}$ *is also independent.*

Recall that independence means: for every finite $I \subseteq T$,

$$\mathbb{P}\left[\bigcap_{i\in I} A_i\right] = \prod_{i\in I} \mathbb{P}[A_i]$$

This can be done in other ways, e.g. by induction (though it's a little bit of a pain) - the proof we'll use involves the Inclusion-Exclusion formula.

Proof. What we want to prove is that

$$\mathbb{P}\left[\bigcap_{i} A_i^c\right] = \prod_{i} \mathbb{P}[A_i^c]$$

given that the $\{A_i\}$ are independent. We start by rewriting

$$\prod_i \mathbb{P}[A_i^c] = \prod_i \big(1 - \mathbb{P}[A_i]\big)$$

$$= 1 - \sum_{\text{all } i} \mathbb{P}[A_i] + \sum_{\text{all } (i,j)} \mathbb{P}[A_i]\mathbb{P}[A_j] - \sum_{\text{all } (i,j,k)} \mathbb{P}[A_i]\mathbb{P}[A_j]\mathbb{P}[A_k] \dots$$

(where "all $(i, j, k, \dots)$" refers only to *unordered* subsets of $[n]$). By independence of $\{A_i\}$ these products are just the probabilities of intersections, so (grouping the sum terms together) the above is

$$1 - \left(\sum_{\text{all } i} \mathbb{P}[A_i] - \sum_{\text{all } (i,j)} \mathbb{P}[A_i \cap A_j] + \sum_{\text{all } (i,j,k)} \mathbb{P}[A_i \cap A_j \cap A_k] \dots\right)$$

But the thing inside the big parens is just the inclusion-exclusion formula! So we get

$$= 1 - \mathbb{P}\left[\bigcup_i A_i\right] = \mathbb{P}\left[\left(\bigcup_i A_i\right)^c\right] = \mathbb{P}\left[\bigcap_i A_i^c\right]$$

and we are done. □

Remark: This technique can also be used to show that changing any subset of the A_i to their complements also preserves independence.

Measuring probability of converging to an average density of x heads

Problem 0.2. *Consider the infinite-coin-toss model ($\Omega = \{0,1\}^\infty$, and σ-algebra $\mathcal{F}$ developed in Lecture 2). Fix some $x \in [0,1]$. Is the set of all sequences whose proportion of 1's converges to x measurable in $\mathcal{F}$?*

Proof. First, we need to define our event. We call it

$$A_x := \Big\{\omega : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \omega_i = x\Big\}$$

To make this easier to work with, we note that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \omega_i = x$ just means "*for all* $m \geq 1$, *there exists* some $N > 0$ (both m, N are integers) such that

$$\text{for all } n \geq N, \quad \frac{1}{n}\sum_{i=1}^{n} \omega_i - x \ \leq \frac{1}{m}$$

We use this to define a collection of sets

$$S_{m,N} := \Big\{\omega : \ \text{for all } n \geq N, \quad \frac{1}{n}\sum_{i=1}^{n} \omega_i - x \ \leq \frac{1}{m}\Big\}$$

Replacing "there exists" and "for all" with their equivalent set operations ($\cup$ and $\cap$ respectively) we get

$$A_x = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} S_{m,N}$$

So if we can show that $S_{m,N} \in \mathcal{F}$ for all m, N, we are done. To do so, let's fix m, N and define for all $k \geq 0$,

$$S_{m,N,k} := \Big\{\omega : \ \text{for } n \in \{N, N+1, \dots, N+k\}, \quad \frac{1}{n}\sum_{i=1}^{n} \omega_i - x \ \leq \frac{1}{m}\Big\}$$

Then we note two facts:

- $S_{m,N} = \bigcap_{k=0}^{\infty} S_{m,N,k}$;
- $S_{m,N,k} \in \mathcal{F}_{N+k} \subset \mathcal{F}_0$ (the algebra from which the σ-algebra $\mathcal{F}$ is built)

These facts together show that $S_{m,N} \in \sigma(\mathcal{F}_0) = \mathcal{F}$, and therefore $A_x \in \mathcal{F}$ as well. □

Of monkeys and typewriters: applying Borel-Cantelli

If you've ever heard the common statement that "a monkey at a typewriter will eventually write the entire works of Shakespeare (infinitely many times, no less)", this is what it really means.

Problem 0.3. *Suppose we have an infinite sequence of random coin flips - so* $\Omega = \{0,1\}^\infty$ *- in which each coin flip is independent and has probability of producing* 1 *("heads") with probability* $p \in (0,1)$. *Let* $b \in \{0,1\}^\ell$ *be any* finite pattern *(so* ℓ *is any positive integer). Prove that, almost surely, the pattern* b *occurs infinitely many times in the sequence.*

To help prove this, we have the Borel-Cantelli lemma:

Proposition 0.1 (**Borel-Cantelli (part 2)**). *Given a sequence* A_n *of events such that (i)* $\sum_n \mathbb{P}[A_n] = \infty$ *and (ii) the events* $\{A_n\}$ *are independent, and defining* $A := \{A_n \text{ i.o.}\}$ *(note: see lecture 3 notes for the definition of this), then* $\mathbb{P}[A] = 1$.

Proof. The intuition is that we break up our outcome ω into disjoint ℓ-length blocks (running from bit $(n-1)\ell + 1$ to bit $n\ell$ so the first block goes from 1 to ℓ); letting b have j zeroes and k ones ($j + k = \ell$), and fixing a particular block $\omega_{((n-1)\ell+1):(n\ell)}$, let A_n be the event that this block is actually equal to b, i.e.

$$A_n = \{\omega : \omega_{((n-1)\ell+1):(n\ell)} = b\}$$

Then, we have

$$\mathbb{P}[A_n] = (1-p)^j p^k > 0 \text{ (because } p \neq 0, 1)$$

Therefore, $\sum_n \mathbb{P}[A_n] = \infty$; furthermore, the events $\{A_n\}$ are independent because the blocks don't overlap. So, almost surely, infinitely many of the A_n come true – and if this happens the sequence b occurs infinitely many times, as we wanted. □

Remark: In reality, I have a hard time believing that a monkey in front of a typewriter will produce a sequence of independent letters, but for the sake of the metaphor we'll pretend that it does.

Lebesgue measure on $\mathbb{R}$

See lecture notes (lecture 2).

EXTRA: Pairwise independence is not independence!

Not covered in recitation, and probably most people have already seen this, but something you should definitely know:

Problem 0.4. *If a collection of events* $\{A_i\}$ *are* pairwise independent *under a probability distribution (i.e. for any* $i \neq j$, $\mathbb{P}[A_i \cap A_j] = \mathbb{P}[A_i]\mathbb{P}[A_j]$*) are they necessarily independent as a collection?*

No, they aren't.

Proof. We'll construct a simple counterexample in the two-fair-coins model ($\Omega = \{0,1\}^2$, $\mathcal{F} = 2^\Omega$, $\mathbb{P}$ uniform). Let "$\oplus$" be the XOR operation, and define:

- $A_1 := \{\omega : \omega_1 = 1\}$;
- $A_2 := \{\omega : \omega_2 = 1\}$;
- $A_\oplus := \{\omega_1 \oplus \omega_2 = 1\}$.

It is easy to check that each event has two elements, and so $\mathbb{P}[A_1] = \mathbb{P}[A_2] = \mathbb{P}[A_\oplus] = 1/2$; it's also easy to check that every pair of events is only satisfied by one elementary outcome (probability $= 1/4$), and so they are pairwise independent.

However, for them to be independent we would need $\mathbb{P}[A_1]\mathbb{P}[A_2]\mathbb{P}[A_\oplus] = \mathbb{P}[A_1 \cap A_2 \cap A_\oplus]$ as well – but the left-hand side is $1/8$ whereas the right-hand side is actually 0 because no event is in all three at once. $\square$