
Final

- You have 180 minutes (9:00-12:00 am) to complete the test.
- This is a closed-book test, except that five $8.5'' \times 11''$ sheets of notes are allowed.
- Calculators are allowed (provided that erasable memory is cleared).
- There are three problems on the quiz. The first is a five-part problem, each part worth 10 points. The second is a six-part problem, each part worth 10 points. There is also an optional seventh part, for which you can receive up to 10 points of extra credit. The third problem consists of four unrelated true-false questions, each worth 10 points.
- The problems are not necessarily in order of difficulty.
- Even if you can't prove a proposition stated in one part of a problem, you may assume that it is true in subsequent parts.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- If we can't read it, we can't grade it.
- If you don't understand a problem, please ask.

Problem F.1 (50 points)

In this problem, we will investigate ML decoding using the standard VA on a minimal conventional trellis, or using a modified VA on a tail-biting trellis.

Consider the $(6, 3, 3)$ binary linear code \mathcal{C} that is generated by the following generator matrix:

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Find the state and branch complexity profiles of an unsectionalized minimal trellis for \mathcal{C} . Draw the corresponding minimal trellis for \mathcal{C} , and label each branch with the corresponding output symbol. Verify that there is a one-to-one map between the codewords of \mathcal{C} and the paths through the trellis.

(b) Show that the following unsectionalized 2-state tail-biting trellis (TBT) realizes \mathcal{C} . (Recall that in a TBT there may be more than one state in $\Sigma_0 = \Sigma_6$, the starting and ending state space, and that the valid paths are those that start and end in the same state in $\Sigma_0 = \Sigma_6$.) Verify that there is a one-to-one map between the codewords of \mathcal{C} and the valid paths through the tail-biting trellis. Draw a normal graph of this tail-biting trellis realization of \mathcal{C} .

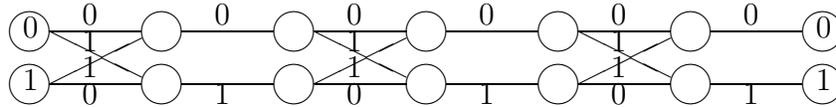


Figure 1. Tail-biting trellis for \mathcal{C} .

(c) Propose a modification to the Viterbi algorithm that finds the maximum-likelihood (ML) codeword in \mathcal{C} , using the tail-biting trellis of part (b). Compare the complexity of your modified VA to that of the standard VA operating on the minimal conventional trellis of part (a).

(d) For a general (n, k) linear code \mathcal{C} with a given coordinate ordering, is it possible to find a minimal unsectionalized tail-biting trellis that simultaneously minimizes the complexity of each of the n state spaces over all tail-biting trellises? Explain.

(e) Given a general linear code \mathcal{C} and an unsectionalized tail-biting trellis for \mathcal{C} , propose a modified VA that performs ML decoding of \mathcal{C} using the tail-biting trellis. In view of the cut-set bound, is it possible to achieve a significant complexity reduction over the standard VA by using such a modified VA?

Problem F.2 (60 points)

In this problem, we will analyze the performance of iterative decoding of a rate-1/3 repeat-accumulate (RA) code on a binary erasure channel (BEC) with erasure probability p , in the limit as the code becomes very long ($n \rightarrow \infty$).

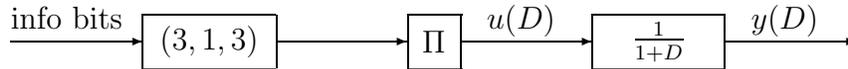


Figure 2. Rate- $\frac{1}{3}$ RA encoder.

The encoder for the rate-1/3 RA code is shown in Figure 2 above, and works as follows. A sequence of information bits is first encoded by an encoder for a (3, 1, 3) repetition code, which simply repeats each information bit three times. The resulting sequence is then permuted by a large pseudo-random permutation Π . The permuted sequence $u(D)$ is then encoded by a rate-1/1 2-state convolutional encoder with input/output relation $y(D) = u(D)/(1 + D)$; *i.e.*, the input/output equation is $y_k = u_k + y_{k-1}$, so the output bit is simply the “accumulation” of all previous input bits (mod 2).

(a) Show that this rate- $\frac{1}{3}$ RA code has the normal graph of Figure 3.

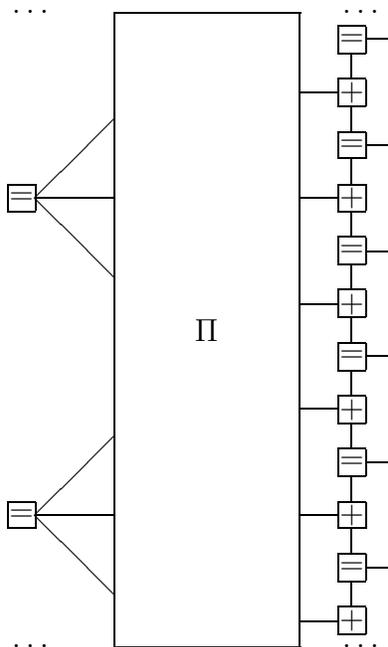


Figure 3. Normal graph of rate- $\frac{1}{3}$ RA code.

(b) Suppose that the encoded bits are sent over a BEC with erasure probability p . Explain how iterative decoding works in this case, using a schedule that alternates between the left constraints and the right constraints.

(c) Show that, as $n \rightarrow \infty$, if the probability that a left-going iterative decoding message is erased is $q_{r \rightarrow \ell}$, then the probability that a right-going message is erased after a left-side update is given by

$$q_{\ell \rightarrow r} = (q_{r \rightarrow \ell})^2.$$

(d) Similarly, show that if the probability that a right-going iterative decoding message is erased is $q_{\ell \rightarrow r}$, then the probability that a left-going message is erased after a right-side update is given by

$$q_{r \rightarrow \ell} = 1 - \frac{(1-p)^2}{(1-p + pq_{\ell \rightarrow r})^2}.$$

[Hint: observe that as $n \rightarrow \infty$, the right-side message probability distributions become invariant to a shift of one time unit.]

(e) Using a version of the area theorem that is appropriate for this scenario, show that iterative decoding cannot succeed if $p \geq \frac{2}{3}$.

(f) The two curves given in parts (c) and (d) are plotted in the EXIT chart below for $p = 0.5$. Show that iterative decoding succeeds in this case.

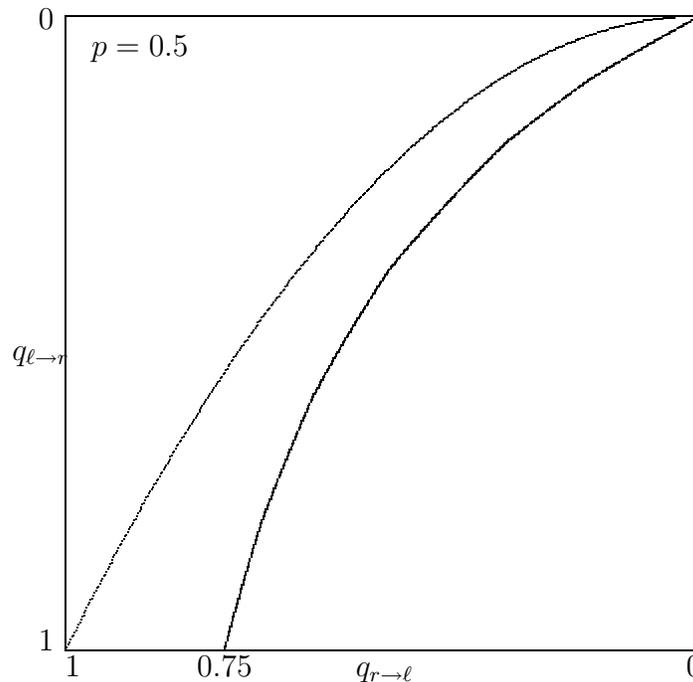


Figure 4. EXIT chart for iterative decoding of a rate- $\frac{1}{3}$ RA code on a BEC with $p = 0.5$.

(g) [Optional; extra credit.] Determine whether or not iterative decoding succeeds for $p = 0.6$.

Problem F.3 (40 points)

For each of the propositions below, state whether the proposition is true or false, and give a proof of not more than a few sentences, or a counterexample. No credit will be given for a correct answer without an adequate explanation.

(a) The Euclidean image of an (n, k, d) binary linear block code is an orthogonal signal set if and only if $k = \log_2 n$ and $d = n/2$.

(b) If a possibly catastrophic binary rate- $1/n$ linear convolutional code with polynomial encoder $\mathbf{g}(D)$ is terminated to a block code $\mathcal{C}_\mu = \{u(D)\mathbf{g}(D) \mid \deg u(D) < \mu\}$, then a set of μ shifts $\{D^k\mathbf{g}(D) \mid 0 \leq k < \mu\}$ of $\mathbf{g}(D)$ is a trellis-oriented generator matrix for \mathcal{C}_μ .

(c) Suppose that a codeword $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in some code \mathcal{C} is sent over a memoryless channel, such that the probability of an output $\mathbf{r} = (r_1, r_2, \dots, r_n)$ is given by $p(\mathbf{r} \mid \mathbf{y}) = \prod_1^n p(r_i \mid y_i)$. Then the *a posteriori* probability $p(Y_1 = y_1, Y_2 = y_2 \mid \mathbf{r})$ is given by

$$p(Y_1 = y_1, Y_2 = y_2 \mid \mathbf{r}) \propto p(Y_1 = y_1 \mid r_1)p(Y_2 = y_2 \mid r_2)p(Y_1 = y_1, Y_2 = y_2 \mid r_3, r_4, \dots, r_n).$$

(d) Let \mathcal{C} be the binary block code whose normal graph is shown in the figure below. All left constraints are repetition constraints and have the same degree d_λ (not counting the dangle); all right constraints have the same degree d_ρ and are given by a binary linear $(d_\rho, \kappa d_\rho)$ constraint code \mathcal{C}_c . Assuming that all constraints are independent, the rate of \mathcal{C} is

$$R = 1 - d_\lambda(1 - \kappa).$$

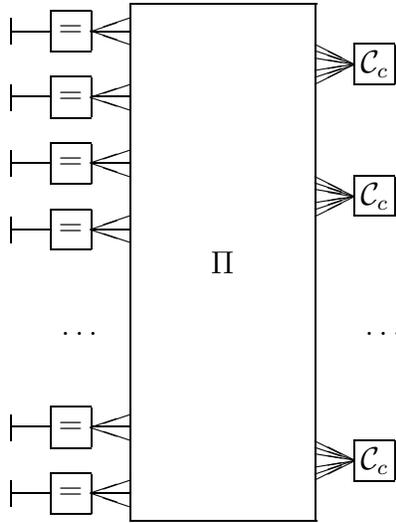


Figure 5. Normal graph realization of \mathcal{C} , with left repetition constraints of degree d_λ and right constraint codes \mathcal{C}_c of degree d_ρ .