Problem Set 1

These exercises use the decibel (dB) scale, defined by:

$$\text{ratio or multiplicative factor of } \alpha \leftrightarrow 10 \log_{10} \alpha \text{ dB}.$$

The following short table should be committed to memory:

α	dB (round numbers)	dB (two decimal places)
1	0	0.00
1.25	1	0.97
2	3	3.01
2.5	4	3.98
e	4.3	4.34
3	4.8	4.77
π	5	4.97
4	6	6.02
5	7	6.99
8	9	9.03
10	10	10.00

Problem 1.1 (Compound interest and dB)

How long does it take to double your money at an interest rate of $P\%$? The bankers' "Rule of 72" estimates that it takes about $72/P$ years; *e.g.,* at a 5% interest rate compounded annually, it takes about 14.4 years to double your money.

(a) An engineer decides to interpolate the dB table above linearly for $1 \leq 1 + p \leq 1.25$; *i.e.,*

$$\text{ratio or multiplicative factor of } 1 + p \leftrightarrow 4p \text{ dB}.$$

Show that this corresponds to a "Rule of 75;" *e.g.,* at a 5% interest rate compounded annually, it takes 15 years to double your money.

(b) A mathematician linearly approximates the dB table for $p \approx 0$ by noting that as $p \to 0$, $\ln(1 + p) \to p$, and translates this into a "Rule of N" for some real number N. What is N? Using this rule, how many years will it take to double your money at a 5% interest rate, compounded annually? What happens if interest is compounded continuously?

(c) How many years will it actually take to double your money at a 5% interest rate, compounded annually? [Hint: $10 \log_{10} 7 = 8.45$ dB.] Whose rule best predicts the correct result?

Problem 1.2 (Biorthogonal codes)

A $2^m \times 2^m$ $\{\pm 1\}$-valued Hadamard matrix H_{2^m} may be constructed recursively as the m-fold tensor product of the 2×2 matrix

$$H_2 = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix},$$

as follows:

$$H_{2^m} = \begin{bmatrix} +H_{2^{m-1}} & +H_{2^{m-1}} \\ +H_{2^{m-1}} & -H_{2^{m-1}} \end{bmatrix}.$$

(a) Show by induction that:

(i) $(H_{2^m})^T = H_{2^m}$, where ^{T} denotes the transpose; *i.e.*, H_{2^m} is symmetric;

(ii) The rows or columns of H_{2^m} form a set of mutually orthogonal vectors of length 2^m;

(iii) The first row and the first column of H_{2^m} consist of all $+1$s;

(iv) There are an equal number of $+1$s and -1s in all other rows and columns of H_{2^m};

(v) $H_{2^m} H_{2^m} = 2^m I_{2^m}$; *i.e.*, $(H_{2^m})^{-1} = 2^{-m} H_{2^m}$, where $^{-1}$ denotes the inverse.

(b) A biorthogonal signal set is a set of real equal-energy orthogonal vectors and their negatives. Show how to construct a biorthogonal signal set of size 64 as a set of $\{\pm 1\}$-valued sequences of length 32.

(c) A simplex signal set S is a set of real equal-energy vectors that are equidistant and that have zero mean $\mathbf{m}(S)$ under an equiprobable distribution. Show how to construct a simplex signal set of size 32 as a set of 32 $\{\pm 1\}$-valued sequences of length 31. [Hint: The fluctuation $O - \mathbf{m}(O)$ of a set O of orthogonal real vectors is a simplex signal set.]

(d) Let $\mathbf{Y} = \mathbf{X} + \mathbf{N}$ be the received sequence on a discrete-time AWGN channel, where the input sequence $\mathbf{X}$ is chosen equiprobably from a biorthogonal signal set B of size 2^{m+1} constructed as in part (b). Show that the following algorithm implements a minimum-distance decoder for B (*i.e.*, given a real 2^m-vector $\mathbf{y}$, it finds the closest $\mathbf{x} \in B$ to $\mathbf{y}$):

(i) Compute $\mathbf{z} = H_{2^m}\mathbf{y}$, where $\mathbf{y}$ is regarded as a column vector;

(ii) Find the component z_j of $\mathbf{z}$ with largest magnitude $|z_j|$;

(iii) Decode to $\mathrm{sgn}(z_j)\mathbf{x}_j$, where $\mathrm{sgn}(z_j)$ is the sign of the largest-magnitude component z_j and $\mathbf{x}_j$ is the corresponding column of H_{2^m}.

(e) Show that a circuit similar to that shown in Figure 1 below for $m = 2$ can implement the $2^m \times 2^m$ matrix multiplication $\mathbf{z} = H_{2^m}\mathbf{y}$ with a total of only $m \times 2^m$ addition and subtraction operations. (This is called the "fast Hadamard transform," or "Walsh transform," or "Green machine.")

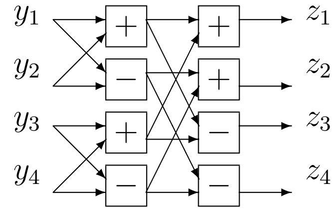

Figure 1. Fast $2^m \times 2^m$ Hadamard transform for $m = 2$.

Problem 1.3 (16-QAM signal sets)

Three 16-point 2-dimensional quadrature amplitude modulation (16-QAM) signal sets are shown in Figure 2, below. The first is a standard 4×4 signal set; the second is the V.29 signal set; the third is based on a hexagonal grid and is the most power-efficient 16-QAM signal set known. The first two have 90° symmetry; the last, only 180°. All have a minimum squared distance between signal points of $d^2_{\min} = 4$.

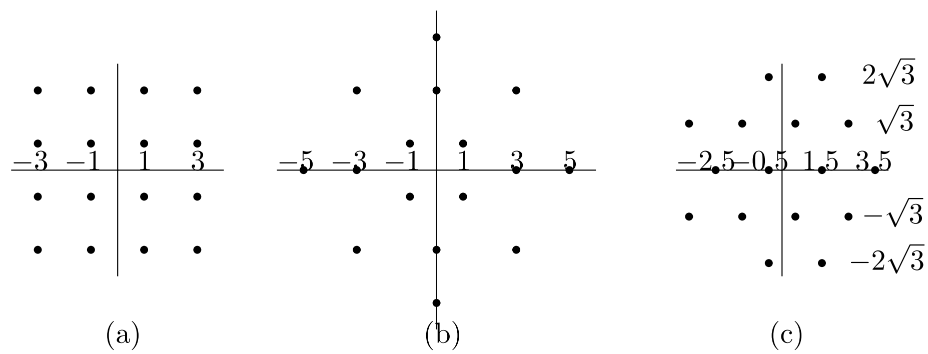

Figure 2. 16-QAM signal sets. (a) (4×4)-QAM; (b) V.29; (c) hexagonal.

(a) Compute the average energy (squared norm) of each signal set if all points are equiprobable. Compare the power efficiencies of the three signal sets in dB.

(b) Sketch the decision regions of a minimum-distance detector for each signal set.

(c) Show that with a phase rotation of $\pm 10°$ the minimum distance from any rotated signal point to any decision region boundary is substantially greatest for the V.29 signal set.

Problem 1.4 (Shaping gain of spherical signal sets)

In this exercise we compare the power efficiency of n-cube and n-sphere signal sets for large n.

An n-cube signal set is the set of all odd-integer sequences of length n within an n-cube of side $2M$ centered on the origin. For example, the signal set of Figure 2(a) is a 2-cube signal set with $M = 4$.

An n-sphere signal set is the set of all odd-integer sequences of length n within an n-sphere of squared radius r^2 centered on the origin. For example, the signal set of Figure

3(a) is also a 2-sphere signal set for any squared radius r^2 in the range $18 \leq r^2 < 25$. In particular, it is a 2-sphere signal set for $r^2 = 64/\pi = 20.37$, where the area πr^2 of the 2-sphere (circle) equals the area $(2M)^2 = 64$ of the 2-cube (square) of the previous paragraph.

Both n-cube and n-sphere signal sets therefore have minimum squared distance between signal points $d^2_{\min} = 4$ (if they are nontrivial), and n-cube decision regions of side 2 and thus volume 2^n associated with each signal point. The point of the following exercise is to compare their average energy using the following large-signal-set approximations:

- The number of signal points is approximately equal to the volume $V(\mathcal{R})$ of the bounding n-cube or n-sphere region $\mathcal{R}$ divided by 2^n, the volume of the decision region associated with each signal point (an n-cube of side 2).
- The average energy of the signal points under an equiprobable distribution is approximately equal to the average energy $E(\mathcal{R})$ of the bounding n-cube or n-sphere region $\mathcal{R}$ under a uniform continuous distribution.

(a) Show that if $\mathcal{R}$ is an n-cube of side $2M$ for some integer M, then under the two above approximations the approximate number of signal points is M^n and the approximate average energy is $nM^2/3$. Show that the first of these two approximations is exact.

(b) For n even, if $\mathcal{R}$ is an n-sphere of radius r, compute the approximate number of signal points and the approximate average energy of an n-sphere signal set, using the following known expressions for the volume $V_\otimes(n, r)$ and the average energy $E_\otimes(n, r)$ of an n-sphere of radius r:

$$\begin{aligned} V_\otimes(n, r) &= \frac{(\pi r^2)^{n/2}}{(n/2)!}; \\ E_\otimes(n, r) &= \frac{nr^2}{n+2}. \end{aligned}$$

(c) For $n = 2$, show that a large 2-sphere signal set has about 0.2 dB smaller average energy than a 2-cube signal set with the same number of signal points.

(d) For $n = 16$, show that a large 16-sphere signal set has about 1 dB smaller average energy than a 16-cube signal set with the same number of signal points. [Hint: $8! = 40320$ (46.06 dB).]

(e) Show that as $n \to \infty$ a large n-sphere signal set has a factor of $\pi e/6$ (1.53 dB) smaller average energy than an n-cube signal set with the same number of signal points. [Hint: Use Stirling's approximation, $m! \to (m/e)^m$ as $m \to \infty$.]