

**Lecture Number 23**

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Jeffrey H. Shapiro

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**Reading:**

- For binary optical communication with squeezed-state light:
    - J.H. Shapiro, H.P. Yuen, and J.A. Machado Mata, “Optical communication with two-photon coherent states—Part II: photoemissive detection and structured receiver performance,” *IEEE Trans. Inform. Theory*, **IT-25**, 179 (1979)
  - For squeezed-state interferometry:
    - C.M. Caves, “Quantum-mechanical noise in an interferometer,” *Phys. Rev. D* **23** 1693 (1981).
    - R.S. Bondurant and J.H. Shapiro, “Squeezed states in phase sensing interferometers,” *Phys. Rev. D* **30**, 2548 (1984).
  - For super-dense coding:
    - M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University, Cambridge, 2000) Sec. 2.3.
  - For quantum lithography:
    - A.N. Boto, P. Kok, D.S. Abrams, S.L. Braunstein, C.P. Williams, and J.P. Dowling, “Quantum interferometric optical lithography: exploiting entanglement to beat the diffraction limit,” *Phys. Rev. Lett.* **85**, 2733 (2000).
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## Introduction

At this point we have completed all the major topics for 6.453. Thus today we will use the time available to survey a variety of applications for non-classical light: binary optical communication with squeezed-state light; squeezed-state interferometry; super-dense coding; and quantum lithography. The squeezed-state applications exploit the signal-to-noise ratio advantage that such non-classical states enjoy in comparison with what is achievable with coherent-state light. Super-dense coding and quantum lithography rely on entangled photons to derive their advantages. Readers who were overwhelmed by the complexity of the continuous-time analyses that we performed in Lectures 21 and 22 should be happy that today's treatments will involve at most two modes at a time. Furthermore, for simplicity, we will only consider idealized lossless conditions in all our examples. We have seen—e.g., in our study of the squeezed-state waveguide tap—that loss can have a disastrous effect on the performance gain afforded by non-classical light. Thus today's material must not be regarded as the final word on the utility of non-classical light in these applications.

## Binary Optical Communication with Squeezed-State Light

Slide 3 shows a simple binary, phase-shift keyed optical communication system that uses a coherent-state transmitter. The transmitter mode,  $\hat{a}e^{-j\omega_0 t}/\sqrt{T}$  for  $0 \leq t \leq T$ , is put into the coherent state  $|\psi_m\rangle$  to encode a single message bit,  $m = 0$  or  $1$ , where

$$|\psi_m\rangle = \begin{cases} |-\sqrt{N}\rangle, & \text{for } m = 0 \\ |\sqrt{N}\rangle, & \text{for } m = 1. \end{cases} \quad (1)$$

That this modulation should be called phase-shift keying is self-evident; the only difference between  $|\psi_0\rangle$  and  $|\psi_1\rangle$  is the  $\pi$  rad phase shift in the coherent-state eigenvalue. We shall assume that the message values  $m = 0$  and  $1$  are equally likely to occur, and that the receiver employs the quantum measurement which minimizes the probability that its decoded bit value,  $\tilde{m} = 0$  or  $1$ , differs from what was transmitted.<sup>1</sup> As noted in the introduction, we will assume a lossless channel, so that the receiver's quantum measurement is made on the  $\hat{a}e^{-j\omega_0 t}/\sqrt{T}$  mode which was excited by the transmitter. The results that we need to determine the optimum quantum receiver and its error probability were already derived on Problem Set 8. However, we will develop some of them here anew in a more general setting.

Suppose that the receiver is confronted with the task of deciding whether the state of the mode associated with the annihilation operator  $\hat{a}$  is given by the density

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<sup>1</sup>Note that we are forcing our receiver to make a decision, i.e., we are not allowing it to make a measurement and, depending on its outcome, say that the data was too noisy to decide without error. See Problem Set 8 for an example of that unambiguous detection approach to binary hypothesis testing.

operator  $\hat{\rho}_0$  (corresponding to  $m = 0$ ), or the density operator  $\hat{\rho}_1$  (corresponding to  $m = 1$ ). Assume that the receiver measures the POVM  $\hat{\Pi}_0, \hat{\Pi}_1$ , where

$$\hat{\Pi}_m^\dagger = \hat{\Pi}_m, \quad \text{for } m = 0, 1 \quad (2)$$

$$\langle \psi | \hat{\Pi}_m | \psi \rangle \geq 0, \quad \text{for } m = 0, 1 \text{ and all } |\psi\rangle \quad (3)$$

$$\hat{\Pi}_0 + \hat{\Pi}_1 = \hat{I}, \quad \text{where } \hat{I} \text{ is the identity operator,} \quad (4)$$

so that the conditional probability that the receiver decides  $\tilde{m} = k$  given  $m = l$  is

$$\Pr(\tilde{m} = k | m = l) = \text{tr}(\hat{\Pi}_k \hat{\rho}_l), \quad \text{for } k = 0, 1 \text{ and } l = 0, 1. \quad (5)$$

Our tasks, therefore, are to choose the POVM to minimize the error probability and then evaluate that optimal performance. These turn out to be straightforward, as we will now show.

The error probability for the preceding receiver satisfies

$$\Pr(e) \equiv \Pr(\tilde{m} \neq m) = \Pr(m = 0) \Pr(\tilde{m} = 1 | m = 0) + \Pr(m = 1) \Pr(\tilde{m} = 0 | m = 1). \quad (6)$$

For equally-likely messages and the receiver we have assumed this result reduces to

$$\Pr(e) = \frac{1}{2} \text{tr}(\hat{\Pi}_1 \hat{\rho}_0) + \frac{1}{2} \text{tr}(\hat{\Pi}_0 \hat{\rho}_1). \quad (7)$$

Using the completeness relation for the POVM, we get

$$\Pr(e) = \frac{1 - \text{tr}[\hat{\Pi}_1(\hat{\rho}_1 - \hat{\rho}_0)]}{2}. \quad (8)$$

The density operator difference,  $\Delta\hat{\rho} \equiv \hat{\rho}_1 - \hat{\rho}_0$ , is Hermitian. Moreover, it has zero trace, i.e.,

$$\text{tr}(\Delta\hat{\rho}) = \text{tr}(\hat{\rho}_1) - \text{tr}(\hat{\rho}_0) = 1 - 1 = 0. \quad (9)$$

Thus, its eigenvalue-eigenket decomposition can be cast in the following form,

$$\Delta\hat{\rho} = \sum_n \rho_n^{(+)} |\rho_n^{(+)}\rangle \langle \rho_n^{(+)}| + \sum_n \rho_n^{(-)} |\rho_n^{(-)}\rangle \langle \rho_n^{(-)}|, \quad (10)$$

where the  $\{\rho_n^{(+)}\}$  are its non-negative eigenvalues, the  $\{\rho_n^{(-)}\}$  are its negative eigenvalues, and  $\{|\rho_n^{(+)}\rangle, |\rho_n^{(-)}\rangle\}$  are its complete orthonormal eigenkets. From Eqs. (8) and (10) we immediately find that

$$\Pr(e) = \frac{1}{2} \left( 1 - \sum_n \rho_n^{(+)} \langle \rho_n^{(+)} | \hat{\Pi}_1 | \rho_n^{(+)} \rangle - \sum_n \rho_n^{(-)} \langle \rho_n^{(-)} | \hat{\Pi}_1 | \rho_n^{(-)} \rangle \right). \quad (11)$$

Now, because  $\hat{\Pi}_1$  is a positive semidefinite operator and because of the algebraic signs of the  $\{\rho_n^{(+)}\}$  and  $\{\rho_n^{(-)}\}$ , we see that

$$\Pr(e) \geq \frac{1}{2} \left( 1 - \sum_n \rho_n^{(+)} \right), \quad (12)$$

with equality when  $\hat{\Pi}_1$  is the projector for the subspace spanned by the  $\{|\rho_n^{(+)}\rangle\}$ , i.e., the non-negative eigenspace of  $\Delta\hat{\rho}$ . At optimality,  $\hat{\Pi}_0$  is therefore the projector for the subspace spanned by the  $\{|\rho_n^{(-)}\rangle\}$ , viz., the negative eigenspace of  $\Delta\hat{\rho}$ . Slide 3 shows an alternative form for this receiver, in which the observable  $\Delta\hat{\rho}$  is measured, yielding a classical outcome  $\Delta\rho$  which is one of the  $\Delta\hat{\rho}$  eigenvalues, and then  $\tilde{m}$  is chosen in accordance with the decision rule,

$$\Delta\rho \underset{\tilde{m}=0}{\overset{\tilde{m}=1}{\gtrless}} 0. \quad (13)$$

The reader should verify that this receiver is indeed equivalent to the POVM receiver given above.

For binary phase-shift keying with coherent-state signals, the conditional density operators are pure-state projectors,

$$\hat{\rho}_0 = |-\sqrt{N}\rangle\langle-\sqrt{N}| \quad \text{and} \quad \hat{\rho}_1 = |\sqrt{N}\rangle\langle\sqrt{N}|. \quad (14)$$

The optimum receiver and its error probability can be computed from the general results we derived in the preceding paragraph, but it is simpler just to employ the work we did on Problem Set 8 to show that

$$\Pr(e)_{\text{CS}} = \frac{1}{2} \left( 1 - \sqrt{1 - |\langle\psi_0|\psi_1\rangle|^2} \right) = \frac{1}{2} \left( 1 - \sqrt{1 - |\langle-\sqrt{N}|\sqrt{N}\rangle|^2} \right) \quad (15)$$

$$= \frac{1}{2} \left( 1 - \sqrt{1 - e^{-4N}} \right) \approx \frac{e^{-4N}}{4}, \quad \text{for } N \gg 1. \quad (16)$$

Now let us reconsider binary phase-shift keying when we use squeezed states instead of coherent states. In this case the message states will be

$$|\psi_0\rangle = |-\beta; \mu, \nu\rangle \quad \text{and} \quad |\psi_1\rangle = |\beta; \mu, \nu\rangle, \quad (17)$$

respectively, where  $\beta, \mu, \nu$  will all be assumed to be positive real. For a fair comparison between the error probability achieved with optimum quantum reception of these squeezed-state signals and what we have already shown for the coherent-state case, we will require that both signal sets use the same average photon number,  $N$ , for their message transmission. In the squeezed-state case this means we require

$$\langle(-1)^{m+1}\beta; \mu, \nu|\hat{a}^\dagger\hat{a}|(-1)^{m+1}\beta; \mu, \nu\rangle = [(\mu - \nu)\beta]^2 + \nu^2 = N, \quad \text{for } m = 0, 1, \quad (18)$$

to match

$$\langle (-1)^{m+1} \sqrt{N} | \hat{a}^\dagger \hat{a} | (-1)^{m+1} \sqrt{N} \rangle = N, \quad \text{for } m = 0, 1, \quad (19)$$

for the coherent-state case.

From Problem Set 8 we have that

$$\Pr(e)_{\text{SS}} = \frac{1}{2} \left( 1 - \sqrt{1 - |\langle \psi_0 | \psi_1 \rangle|^2} \right) = \frac{1}{2} \left( 1 - \sqrt{1 - |\langle -\beta; \mu, \nu | \beta; \mu, \nu \rangle|^2} \right) \quad (20)$$

$$= \frac{1}{2} \left( 1 - \sqrt{1 - e^{-4\beta^2}} \right), \quad (21)$$

which we should minimize by choice of  $\beta, \mu, \nu$  subject to the average photon number constraint (18). We have already done that optimization—in a different setting—on Problem Set 5. There we showed that

$$\beta = \sqrt{N(N+1)}, \quad \mu = (N+1)/\sqrt{2N+1}, \quad \nu = N/\sqrt{2N+1} \quad (22)$$

satisfied  $\langle \hat{a}^\dagger \hat{a} \rangle = N$  and maximized the homodyne signal-to-noise ratio,

$$\text{SNR}_{\text{hom}} \equiv \frac{\langle \hat{a}_1 \rangle^2}{\langle \Delta \hat{a}_1^2 \rangle}, \quad (23)$$

where  $\hat{a}_1 \equiv \text{Re}(\hat{a})$ . But, for the squeezed state  $|\beta; \mu, \nu\rangle$  with  $\beta, \mu, \nu > 0$  we have that

$$\langle \hat{a}_1 \rangle = (\mu - \nu)\beta \quad \text{and} \quad \langle \Delta \hat{a}_1^2 \rangle = (\mu - \nu)^2/4. \quad (24)$$

Thus the homodyne SNR for this squeezed state is  $\text{SNR} = 4\beta^2$ , and therefore Eq. (22) provides the squeezed-state parameters that minimize the error probability expression in Eq. (21). So, for optimum squeezed-state binary phase-shift keying the error probability is

$$\Pr(e)_{\text{SS}} = \frac{1}{2} \left( 1 - \sqrt{1 - e^{-4N(N+1)}} \right) \approx \frac{e^{-4N^2}}{4}, \quad \text{for } N \gg 1. \quad (25)$$

Comparing this result to what we have previously found for the coherent-state transmitter reveals an enormous reduction in the error probability. Remember, however, that we assumed lossless operation. The SNR degradation arising from loss that we encountered in the squeezed-state waveguide tap will also affect the error probability calculation for binary phase-shift keying with squeezed-state light when there is loss. We shall not bother to work out the details.

## Squeezed-State Interferometry

Slide 5 shows a phase-conjugate Mach-Zehnder interferometer that uses coherent-state light. Two input modes, with annihilation operators  $\hat{a}_{\text{in}}$  and  $\hat{b}_{\text{in}}$ , enter a 50/50 beam splitter, producing internal modes with annihilation operators

$$\hat{a} = \frac{\hat{a}_{\text{in}} + \hat{b}_{\text{in}}}{\sqrt{2}} \quad \text{and} \quad \hat{b} = \frac{\hat{a}_{\text{in}} - \hat{b}_{\text{in}}}{\sqrt{2}}. \quad (26)$$

These internal modes encounter phase shifts  $\pm\phi$  en route to another 50/50 beam splitter, so that the annihilation operators for the interferometer's output modes are

$$\hat{a}_{\text{out}} = \frac{\hat{a}e^{j\phi} + \hat{b}e^{-j\phi}}{\sqrt{2}} = \hat{a}_{\text{in}} \cos(\phi) + j\hat{b}_{\text{in}} \sin(\phi) \quad (27)$$

$$\hat{b}_{\text{out}} = \frac{-\hat{a}e^{j\phi} + \hat{b}e^{-j\phi}}{\sqrt{2}} = -j\hat{a}_{\text{in}} \sin(\phi) - \hat{b}_{\text{in}} \cos(\phi). \quad (28)$$

The purpose of the interferometer is to measure a very small phase shift,  $|\phi| \ll 1$ , by means of photodetection measurements on one or both of its output modes. When the  $\hat{a}_{\text{in}}$  mode is in the coherent state  $|\sqrt{N}\rangle$  and the  $\hat{b}_{\text{in}}$  mode is in its vacuum state  $|0\rangle$ , we find that

$$\langle \hat{a}_{\text{out}} \rangle = \sqrt{N} \cos(\phi) \approx \sqrt{N}, \quad \text{for } |\phi| \ll 1 \quad (29)$$

$$\langle \hat{b}_{\text{out}} \rangle = -j\sqrt{N} \sin(\phi) \approx -j\sqrt{N}\phi, \quad \text{for } |\phi| \ll 1. \quad (30)$$

Consequently, we shall *estimate* the phase shift  $\phi$  by using (unity quantum efficiency) homodyne detection to measure  $\hat{b}_{\text{out}_2} \equiv \text{Im}(\hat{b}_{\text{out}})$  and then take our phase estimate to be<sup>2</sup>

$$\tilde{\phi} \equiv -\hat{b}_{\text{out}_2}/\sqrt{N}. \quad (31)$$

The average value of this estimate equals the unknown phase, i.e.,

$$\langle \tilde{\phi} \rangle = -\langle \hat{b}_{\text{out}_2} \rangle/\sqrt{N} = \phi, \quad (32)$$

so the mean value of the estimation error,  $\Delta\tilde{\phi} \equiv \phi - \tilde{\phi}$ , is zero. The mean-squared error of this estimate is therefore

$$\langle \Delta\tilde{\phi}^2 \rangle = \langle \Delta\hat{b}_{\text{out}_2}^2 \rangle/N = 1/4N, \quad (33)$$

where the second equality follows because the  $\hat{b}_{\text{out}}$  mode is in a coherent state.<sup>3</sup>

Equation (33) is referred to as the standard quantum limit (SQL) for this phase-conjugate interferometer. Because this system uses classical light, the preceding estimation performance could have been calculated—and the same results obtained—by means of semiclassical photodetection theory. Hence Eq. (33) is also known as the shot-noise limit. As a prelude to squeezed-state interferometry, let us consider the physical origin of the noise in the estimate  $\tilde{\phi}$ . In semiclassical theory that noise is

<sup>2</sup>We are being a bit fast and loose here with notation. The left-hand side of this equation is a classical random variable,  $\tilde{\phi}$ , that is our estimate of  $\phi$ . The right-hand side of this equation is a Hilbert-space operator. What we are saying, of course, is that  $\tilde{\phi}$  is a classical random variable whose statistics coincide with those of the operator  $-\hat{b}_{\text{out}_2}/\sqrt{N}$ .

<sup>3</sup>Equations (27) and (28) are a beam-splitter relation whose input modes are in coherent states. We know, from the homework, that the output modes will then also be in coherent states.

local-oscillator shot noise, and there is scarce little that we can do about it. But we know that the semiclassical theory of photodetection is quantitatively correct, for coherent-state light, but nonetheless qualitatively wrong. The noise in balanced homodyne detection—regardless of whether the signal beam being measured is in a classical or a non-classical state—is the quantum noise in the measured quadrature of that signal beam. From Eq. (28) we can show that

$$\langle \Delta \hat{b}_{\text{out}_2}^2 \rangle = \langle \Delta \hat{a}_{\text{in}_1}^2 \rangle \sin^2(\phi) + \langle \Delta \hat{b}_{\text{in}_2}^2 \rangle \cos^2(\phi) \approx \langle \Delta \hat{a}_{\text{in}_1}^2 \rangle \phi^2 + \langle \Delta \hat{b}_{\text{in}_2}^2 \rangle, \quad \text{for } |\phi| \ll 1, \quad (34)$$

when the  $\hat{a}_{\text{in}}$  and  $\hat{b}_{\text{in}}$  modes are in a product state, such as their each being in a coherent state. Just as we saw in the squeezed-state waveguide tap, we have the opportunity to improve performance—reduce noise in the quadrature we will measure—by changing a vacuum-state mode into a squeezed-vacuum state mode, i.e., put the  $\hat{b}_{\text{in}}$  mode into a squeezed-vacuum state. While we are at it, we might as well let the  $\hat{a}_{\text{in}}$  mode be in a squeezed state too.

The setup we will consider for squeezed-state interferometry, shown on slide 6, has the same phase-conjugate Mach-Zehnder structure as the coherent-state system we have already evaluated. Now, however, the  $\hat{a}_{\text{in}}$  mode is in the squeezed state  $|\beta; \mu, \nu\rangle$  and the  $\hat{b}_{\text{in}}$  mode is in the squeezed-vacuum state  $|0; \mu, -\nu\rangle$ , with  $\beta, \mu, \nu > 0$ . It is a simple matter to verify that

$$\langle \hat{a}_{\text{out}} \rangle = (\mu - \nu)\beta \cos(\phi) \approx (\mu - \nu)\beta, \quad \text{for } |\phi| \ll 1 \quad (35)$$

$$\langle \hat{b}_{\text{out}} \rangle = -j(\mu - \nu)\beta \sin(\phi) \approx -j(\mu - \nu)\beta\phi, \quad \text{for } |\phi| \ll 1. \quad (36)$$

Hence we will use

$$\tilde{\phi} \equiv -\hat{b}_{\text{out}_2} / [(\mu - \nu)\beta] \quad (37)$$

as our estimate of the unknown phase  $\phi$ , from which it follows that

$$\langle \tilde{\phi} \rangle = \phi \quad (38)$$

and

$$\langle \Delta \tilde{\phi}^2 \rangle = \frac{(\mu - \nu)^2 \phi^2 + (\mu - \nu)^2}{4[(\mu - \nu)\beta]^2} \approx 1/4\beta^2, \quad \text{for } |\phi| \ll 1. \quad (39)$$

It is tempting to apply the same  $4\beta^2$  optimization that we used in the binary phase-shift keying application to minimize this mean-squared phase estimation error, but that is not an appropriate thing to do. For a fair comparison between the squeezed-state interferometer and the coherent-state interferometer, each one must employ the same average number of photons. For the coherent-state system, all the photons enter the interferometer via the  $\hat{a}_{\text{in}}$  mode, and we have

$$\langle \hat{a}^\dagger \hat{a}_{\text{in}} \rangle + \langle \hat{b}_{\text{in}}^\dagger \hat{b}_{\text{in}} \rangle = N + 0 = N. \quad (40)$$

For the squeezed-state interferometer, on the other hand, *both* input modes are excited, and we get

$$\langle \hat{a}^\dagger \hat{a}_{\text{in}} \rangle + \langle \hat{b}_{\text{in}}^\dagger \hat{b}_{\text{in}} \rangle = \{[(\mu - \nu)\beta]^2 + \nu^2\} + \nu^2 = [(\mu - \nu)\beta]^2 + 2\nu^2 = N, \quad (41)$$

where the last equality is our average photon-number constraint on  $\beta, \mu, \nu$ . So, we must maximize  $\beta^2$  subject to this constraint. A little calculus yields

$$\beta = \sqrt{N(N+2)/2}, \quad \mu = \frac{N+2}{2\sqrt{N+1}}, \quad \nu = \frac{N}{2\sqrt{N+1}}, \quad (42)$$

as the optimum squeezed state for this application, which, in turn, gives

$$\langle \Delta \tilde{\phi}^2 \rangle = 1/2N(N+2) \approx 1/2N^2, \quad \text{for } N \gg 1. \quad (43)$$

for the mean-squared phase estimation error.

The performance of the optimized squeezed-state interferometer is much better than that of its coherent-state counterpart. Once again, however, we have assumed lossless conditions. When there is loss in the interferometer's arms or sub-unity quantum efficiency in the homodyne detector, this performance advantage will be degraded. Here too we shall not bother working out the details.

## Super-Dense Coding

Let us return to binary optical communication in an ideal lossless setting. This time, however, we shall employ non-classical states throughout, and show an advantage that derives from using entangled states instead of single-photon states. We will start with single-photon binary polarization modulation, as shown on slide 7. Here, a message  $m$ —equally likely to be 0 or 1—is sent by exciting the field modes  $(\hat{a}_x \vec{i}_x + \hat{a}_y \vec{i}_y)e^{-j\omega_0 t}/\sqrt{T}$  for  $0 \leq t \leq T$  in the horizontally polarized single-photon state,  $|\psi_0\rangle = |H\rangle$ , when  $m = 0$  and in the vertically polarized single-photon state,  $|\psi_1\rangle = |V\rangle$ , when  $m = 1$ . It is easy to see that the ideal polarization analysis system shown on slide 7 allows the receiver to make an error-free determination of which message was sent. Because there is no loss in the system, the transmitter's single-photon state arrives unimpeded at the receiver. There, it is transmitted by the polarizing beam splitter if that photon is horizontally polarized, and it is reflected if that photon is vertically polarized. Ideal photon counters placed at the output ports of the polarizing beam splitter then provide perfect discrimination between these two message possibilities.<sup>4</sup> Thus, this system achieves error-free communication of a single binary digit (bit)

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<sup>4</sup>In terms of the optimum quantum measurement theory that we developed earlier in this lecture we know that  $\Delta \hat{\rho} = (|V\rangle\langle V| - |H\rangle\langle H|)/2$  has  $|V\rangle$  spanning its non-negative eigenspace and  $|H\rangle$  spanning its negative eigenspace within the Hilbert space of single photon states for the modes associated with  $\{\hat{a}_x, \hat{a}_y\}$ . So, the polarization analysis system does realize the optimum quantum measurement.

by transmission of one photon. Now let us see how entanglement can increase the communication capacity to two bits per photon.

Suppose that Alice and Bob have shared a singlet state of two photons,

$$|\psi^-\rangle_{AB} = \frac{|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B}{\sqrt{2}}, \quad (44)$$

in advance of attempting any information transmission. Alice wants to send *two* bits of information— $m = 00, 01, 10, \text{ or } 11$ —to Bob by transmission of her single photon. To do so, she applies one of four distinct waveplate transformations to her half of the singlet state to encode the message  $m$ . Assuming<sup>5</sup> that her photon was initially in the state  $\alpha|H\rangle_A + \beta|V\rangle_A$ , where  $|\alpha|^2 + |\beta|^2 = 1$ , these waveplate transformations are:

$$\alpha|H\rangle_A + \beta|V\rangle_A \longrightarrow \alpha|H\rangle_A + \beta|V\rangle_A, \quad \text{if } m = 00 \quad (45)$$

$$\alpha|H\rangle_A + \beta|V\rangle_A \longrightarrow \alpha|H\rangle_A - \beta|V\rangle_A, \quad \text{if } m = 01 \quad (46)$$

$$\alpha|H\rangle_A + \beta|V\rangle_A \longrightarrow \alpha|V\rangle_A + \beta|H\rangle_A, \quad \text{if } m = 10 \quad (47)$$

$$\alpha|H\rangle_A + \beta|V\rangle_A \longrightarrow \alpha|V\rangle_A - \beta|H\rangle_A, \quad \text{if } m = 11. \quad (48)$$

Viewed on the tensor-product Hilbert space for the joint state of Alice and Bob's photons, Alice's waveplate transformation changes  $|\psi^-\rangle_{AB}$  into one of the following message-bearing states,

$$|\psi_{00}\rangle_{AB} = \frac{|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B}{\sqrt{2}}, \quad \text{if } m = 00 \quad (49)$$

$$|\psi_{01}\rangle_{AB} = \frac{|H\rangle_A|V\rangle_B + |V\rangle_A|H\rangle_B}{\sqrt{2}}, \quad \text{if } m = 01 \quad (50)$$

$$|\psi_{10}\rangle_{AB} = \frac{|H\rangle_A|H\rangle_B - |V\rangle_A|V\rangle_B}{\sqrt{2}}, \quad \text{if } m = 10 \quad (51)$$

$$|\psi_{11}\rangle_{AB} = \frac{|H\rangle_A|H\rangle_B + |V\rangle_A|V\rangle_B}{\sqrt{2}}, \quad \text{if } m = 11, \quad (52)$$

which we recognize—from our work on qubit teleportation—as the four Bell states. So, when Alice transmits her modulated photon to Bob, he now has *both* halves of a Bell state. And, because the Bell states are orthonormal, he can measure the POVM  $\{\hat{\Pi}_m : m = 00, 01, 10, 11\}$ , with

$$\hat{\Pi}_m \equiv |\psi_m\rangle_{ABAB}\langle\psi_m|, \quad \text{for } m = 00, 01, 10, 11, \quad (53)$$

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<sup>5</sup>As we know, Alice's half of the singlet state is in a completely mixed state,  $\hat{\rho}_A = (|H\rangle_{AA}\langle H| + |V\rangle_{AA}\langle V|)/2$ , but this pure-state assumption makes it possible for us to provide an explicit description of the waveplate transformations.

and receive Alice's two-bit message without error. If we discount Alice and Bob's initial sharing of the singlet state, which did not in itself convey any information between them, then Alice has just sent Bob two bits of information by transmission of her one photon. In comparison with the previous single-photon polarization system, this entanglement-based approach increases communication capacity by a factor of two. Because communication of one bit is the ultimate limit for one-time transmission using an unentangled single photon, we can say that the unentangled system on slide 7 uses dense coding. For that reason, the entangled system that we have just analyzed is said to use super-dense coding: it communicates two bits of information by means of one-time transmission of a single-photon state.

## Quantum Lithography

Our last example, quantum lithography, is also based on entanglement. Consider the setup shown on slide 10. Here, we are trying to lay down a sinusoidal fringe pattern on a photoresist (located in the  $z = 0$  plane) so that, after processing, we can obtain a mask for producing that pattern on a semiconductor wafer.<sup>6</sup> The slide 10 setup uses coherent-state light, i.e., the plane-wave field modes

$$\frac{\hat{a}_+ e^{-j(\omega t - \vec{k}_+ \cdot \vec{r})}}{\sqrt{T}} \quad \text{and} \quad \frac{\hat{a}_- e^{-j(\omega t - \vec{k}_- \cdot \vec{r})}}{\sqrt{T}} \quad \text{for } 0 \leq t \leq T, \quad (54)$$

are both in their  $|\sqrt{N/2}\rangle$  coherent states. Their propagation vectors obey

$$\vec{k}_\pm = \pm k \sin(\theta) \vec{i}_x + k \cos(\theta) \vec{i}_z, \quad (55)$$

where  $k \equiv \omega/c$ . On the surface of the photoresist the total field operator is therefore

$$\hat{E}(x, t) = \frac{\hat{a}_+ e^{-j(\omega t - k \sin(\theta)x)}}{\sqrt{T}} + \frac{\hat{a}_- e^{-j(\omega t + k \sin(\theta)x)}}{\sqrt{T}}, \quad \text{for } 0 \leq t \leq T, \quad (56)$$

where we have neglected vacuum-state modes that are needed to give  $\hat{E}$  the proper free-field commutator bracket with its adjoint.<sup>7</sup> We will assume that the photoresist is illuminated for the full  $0 \leq t \leq T$  time interval, resulting in an exposure that is proportional to

$$\int_0^T dt \hat{E}^\dagger(x, t) \hat{E}(x, t) = \hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_- + 2\text{Re}(\hat{a}_+ \hat{a}_-^\dagger e^{2jk \sin(\theta)x}). \quad (57)$$

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<sup>6</sup>Producing a fringe pattern is the most elementary photolithographic task. For VLSI fabrication, much more complicated patterns are needed. Nevertheless, to illustrate the basic idea of quantum lithography, we will limit our consideration to this simple example.

<sup>7</sup>These vacuum-state modes will not contribute to the measurement made by the photoresist, so it is safe to ignore them.

For the  $\hat{a}_\pm$  modes in the assumed coherent states, we have that the average value of the exposure is

$$\left\langle \int_0^T dt \hat{E}^\dagger(x, t) \hat{E}(x, t) \right\rangle = \langle \hat{a}_+^\dagger \hat{a}_+ \rangle + \langle \hat{a}_-^\dagger \hat{a}_- \rangle + 2\text{Re}(\langle \hat{a}_+ \rangle \langle \hat{a}_-^\dagger \rangle e^{2jk \sin(\theta)x}) \quad (58)$$

$$= N[1 + \cos(2k \sin(\theta)x)]. \quad (59)$$

This is a sinusoidal fringe pattern,  $\mathcal{P}(x)$ , with 100% visibility, i.e.,

$$\frac{\max[\mathcal{P}(x)] - \min[\mathcal{P}(x)]}{\max[\mathcal{P}(x)] + \min[\mathcal{P}(x)]} = 1. \quad (60)$$

It has spatial period  $\lambda/2 \sin(\theta)$ , with  $\lambda = 2\pi/k$  being the wavelength of the illumination. Thus, for coherent-state illumination at wavelength  $\lambda$ , the finest sinusoidal fringe pattern that we can produce has period  $\lambda/2$ . It is realized as the illumination approaches grazing incidence on the photoresist, i.e., as  $\theta \rightarrow \pi/2$ .

Now consider the quantum lithography setup shown on slide 11. It differs from the coherent-state system we have just discussed in only two ways. First, the  $\hat{a}_\pm$  modes are in the entangled state

$$|N00N\rangle \equiv \frac{|N\rangle_+ |0\rangle_- + |0\rangle_+ |N\rangle_-}{\sqrt{2}}, \quad (61)$$

where  $|N\rangle_\pm$  are the  $N$ -photon states of the  $\hat{a}_\pm$  modes and  $|0\rangle_\pm$  are their vacuum states. Second, the resist is an  $N$ -photon absorber, i.e., it responds to

$$\int_0^T dt K \hat{E}^{\dagger N}(x, t) \hat{E}^N(x, t),$$

for some positive constant  $K$ . The average value of this quantity then turns out to be

$$\left\langle \int_0^T dt K \hat{E}^{\dagger N}(x, t) \hat{E}^N(x, t) \right\rangle = KN![1 + \cos(2kN \sin(\theta)x)]/T^{N-1}, \quad (62)$$

as the reader may want to verify. This average exposure is a 100% visibility sinusoidal fringe pattern whose spatial period is  $\lambda/2N \sin(\theta)$ . Entanglement of  $N$  photons has enabled a factor-of- $N$  reduction in the spatial period *without* changing the wavelength of the light.

Preliminary proof-of-principle demonstrations—using photodetectors instead of photoresists—have been reported for  $N00N$ -state generation. There are, however, substantial issues to be confronted in obtaining  $N$ -photon photoresists. Nevertheless, quantum lithography provides another example of the improved capabilities that can be obtained by use of non-classical illumination, albeit one that needs considerable more research before it can be brought to fruition.

## **The End of the Road...**

We have now reached the end of the lecture notes for 6.453, but it is not the end of the road. With this foundation, plus some collateral knowledge, the reader should be well prepared to enter research in quantum optical communication. For example, knowledge of classical information theory, plus what we have covered in 6.453, is sufficient to begin research on the classical information capacity of optical communication channels in a fully quantum setting. Likewise, knowledge of basic Fourier optics, plus what we have covered in 6.453, is sufficient to begin research on quantum imaging, i.e., imaging systems that employ non-classical resources to obtain improved resolution.

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