Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 6
Fall 2016

Issued: Thursday, October 13, 2016 **Due:** Thursday, October 20, 2016

Problem 6.1

Here we begin the analysis of quantum linear transformations by treating the single-frequency quantum theory of the beam splitter. Consider the arrangement shown in Fig. 1. Here, $\hat{a}_{\text{IN}}$ and $\hat{b}_{\text{IN}}$ are the annihilation operators of the frequency-ω components of the quantum fields entering the two input ports of the beam splitter, and $\hat{a}_{\text{OUT}}$ and $\hat{b}_{\text{OUT}}$ are the corresponding frequency-ω annihilation operators at the two output ports. The input-output relation for this beam splitter is the following:

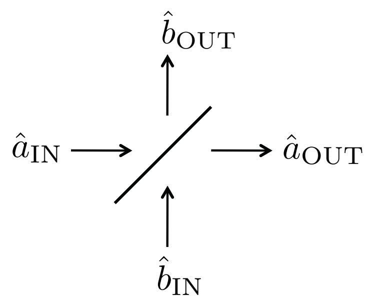

Figure 1: Single-frequency beam splitter configuration

$$
\begin{aligned}
\hat{a}_{\text{OUT}} &= \sqrt{\epsilon}\,\hat{a}_{\text{IN}} + \sqrt{1-\epsilon}\,\hat{b}_{\text{IN}} \\
\hat{b}_{\text{OUT}} &= -\sqrt{1-\epsilon}\,\hat{a}_{\text{IN}} + \sqrt{\epsilon}\,\hat{b}_{\text{IN}},
\end{aligned}
$$

where $0 < \epsilon < 1$ is the power-transmission of the beam splitter, i.e., the fraction of the incident photon flux that passes straight through the device (from $\hat{a}_{\text{IN}}$ to $\hat{a}_{\text{OUT}}$ or from $\hat{b}_{\text{IN}}$ to $\hat{b}_{\text{OUT}}$).

(a) Show that the beam splitter's input-output relation is lossless, i.e., prove that

$$\hat{a}^\dagger_{\text{OUT}}\hat{a}_{\text{OUT}} + \hat{b}^\dagger_{\text{OUT}}\hat{b}_{\text{OUT}} = \hat{a}^\dagger_{\text{IN}}\hat{a}_{\text{IN}} + \hat{b}^\dagger_{\text{IN}}\hat{b}_{\text{IN}},$$

so that regardless of the joint state of the $\hat{a}_{\text{IN}}$ and $\hat{b}_{\text{IN}}$ modes, the total photon number in the output modes is the same as the total photon number in the input modes.

(b) The inputs to the beam splitter have the usual commutators for annihilation operators of independent modes:

$$\begin{aligned}[\hat{a}_{\rm IN}, \hat{b}_{\rm IN}] &= [\hat{a}_{\rm IN}, \hat{b}^\dagger_{\rm IN}] = 0 \\ [\hat{a}_{\rm IN}, \hat{a}^\dagger_{\rm IN}] &= [\hat{b}_{\rm IN}, \hat{b}^\dagger_{\rm IN}] = 1.\end{aligned}$$

Show that the beam splitter's input-output relation is commutator preserving, i.e., prove that

$$\begin{aligned}[\hat{a}_{\rm OUT}, \hat{b}_{\rm OUT}] &= [\hat{a}_{\rm OUT}, \hat{b}^\dagger_{\rm OUT}] = 0 \\ [\hat{a}_{\rm OUT}, \hat{a}^\dagger_{\rm OUT}] &= [\hat{b}_{\rm OUT}, \hat{b}^\dagger_{\rm OUT}] = 1.\end{aligned}$$

(c) The joint state of the input modes, $\hat{a}_{\rm IN}$ and $\hat{b}_{\rm IN}$, is their density operator, $\hat{\rho}_{\rm IN}$. This density operator is fully characterized by its normally-ordered form,

$$\rho^{(n)}_{\rm IN}(\alpha^*, \beta^*; \alpha, \beta) \equiv {}_{\rm IN}\langle\beta|_{\rm IN}\langle\alpha|\hat{\rho}_{\rm IN}|\alpha\rangle_{\rm IN}|\beta\rangle_{\rm IN},$$

where $|\alpha\rangle_{\rm IN}$ and $|\beta\rangle_{\rm IN}$ are the coherent states of the $\hat{a}_{\rm IN}$ and $\hat{b}_{\rm IN}$ modes. The 4-D Fourier transform of $\rho^{(n)}_{\rm IN}(\alpha^*, \beta^*; \alpha, \beta)$ is then the anti-normally ordered joint characteristic function,

$$\chi^{\rho_{\rm IN}}_A(\zeta^*_a, \zeta^*_b; \zeta_a, \zeta_b) \equiv {\rm tr}\left(\hat{\rho}_{\rm IN} e^{-\zeta^*_a \hat{a}_{\rm IN} - \zeta^*_b \hat{b}_{\rm IN}} e^{\zeta_a \hat{a}^\dagger_{\rm IN} + \zeta_b \hat{b}^\dagger_{\rm IN}}\right),$$

where ζ_a and ζ_b are complex numbers. Relate the anti-normally ordered characteristic function of the output modes,

$$\chi^{\rho_{\rm OUT}}_A(\zeta^*_a, \zeta^*_b; \zeta_a, \zeta_b) \equiv {\rm tr}\left(\hat{\rho}_{\rm OUT} e^{-\zeta^*_a \hat{a}_{\rm OUT} - \zeta^*_b \hat{b}_{\rm OUT}} e^{\zeta_a \hat{a}^\dagger_{\rm OUT} + \zeta_b \hat{b}^\dagger_{\rm OUT}}\right),$$

to that for the input modes by: (1) using the beam splitter's input-output relation to write the exponential terms in the $\chi^{\rho_{\rm OUT}}_A(\zeta^*_a, \zeta^*_b; \zeta_a, \zeta_b)$ definition in terms of the input-mode annihilation and creation operators, and (2) evaluating the expectation of the product of the resulting exponential terms by multiplying by the joint density operator of the input modes and taking the trace.

(d) Suppose that the joint state of $\hat{a}_{\rm IN}$ and $\hat{b}_{\rm IN}$ is the two-mode coherent state $|\alpha_{\rm IN}\rangle_{\rm IN}|\beta_{\rm IN}\rangle_{\rm IN}$. Use the result of (c) to show that the joint state of $\hat{a}_{\rm OUT}$ and $\hat{b}_{\rm OUT}$ is the two-mode coherent state $|\alpha_{\rm OUT}\rangle_{\rm OUT}|\beta_{\rm OUT}\rangle_{\rm OUT}$ where

$$\begin{aligned}\alpha_{\rm OUT} &= \sqrt{\epsilon}\,\alpha_{\rm IN} + \sqrt{1-\epsilon}\,\beta_{\rm IN}, \\ \beta_{\rm OUT} &= -\sqrt{1-\epsilon}\,\alpha_{\rm IN} + \sqrt{\epsilon}\,\beta_{\rm IN}.\end{aligned}$$

Problem 6.2

Here we shall develop a moment-generating function approach to the quantum statistics of single-mode direct detection. Suppose that an ideal photodetector is used to make the number-operator measurement, $\hat{N} \equiv \hat{a}^\dagger \hat{a}$, on a single-mode field whose state is given by the density operator $\hat{\rho}$ and let N denote the classical random variable outcome of this quantum measurement. The moment-generating function of N is

$$M_N(s) \equiv \sum_{n=0}^{\infty} e^{sn}\,\Pr(N=n) = \sum_{n=0}^{\infty} e^{sn}\langle n|\hat{\rho}|n\rangle, \quad \text{for } s \text{ real}, \tag{1}$$

where the second equality follows from Problem 3.2(b). (The moment-generating function of a random variable, from classical probability theory, is the Laplace transform of the probability density function of that random variable—cf. the characteristic function, which is the Fourier transform of the probability density—and hence provides a complete characterization of the random variable. In other words, the probability density function can be recovered from the moment-generating function by an inverse Laplace transform operation.)

(a) Define a function $Q_N(\lambda)$ as follows,

$$Q_N(\lambda) = \sum_{n=0}^{\infty} (1-\lambda)^n \langle n|\hat{\rho}|n\rangle, \quad \text{for } \lambda \text{ real}. \tag{2}$$

Show how $M_N(s)$ can be found from $Q_N(\lambda)$.

(b) Show that

$$\begin{aligned} \left.\frac{d^k Q_N(\lambda)}{d\lambda^k}\right|_{\lambda=0} &= \sum_{n=k}^{\infty} (-1)^k n(n-1)(n-2)\cdots(n-k+1)\langle n|\hat{\rho}|n\rangle \\ &= (-1)^k \langle \hat{a}^{\dagger k}\hat{a}^k\rangle, \quad \text{for } k = 1,2,3\ldots \end{aligned}$$

(The last equalilty explains why $\langle \hat{a}^{\dagger k}\hat{a}^k\rangle$ is called the kth factorial moment of the photon count.)

(c) Suppose that $\hat{\rho} = |m\rangle\langle m|$, i.e., that the field mode is in the mth number state. Find the factorial moments $\{\,\langle \hat{a}^{\dagger k}\hat{a}^k\rangle : k = 1,2,3,\ldots\}$. Use the Taylor series,

$$Q_N(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\left.\frac{d^k Q_N(\lambda)}{d\lambda^k}\right|_{\lambda=0}\right)\lambda^k$$

to find $Q_N(\lambda)$ and then use the result of part (a) to find $M_N(s)$. Verify that this moment-generating function agrees with what you would find directly from Eq. (1).

(d) Suppose that $\hat{\rho} = |\alpha\rangle\langle\alpha|$, i.e., that the field mode is in a coherent state with eigenvalue α. Find the factorial moments $\{\langle \hat{a}^{\dagger k}\hat{a}^k\rangle : k = 1, 2, 3, \ldots\}$. Use the Taylor series,

$$Q_N(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\left.\frac{d^k Q_N(\lambda)}{d\lambda^k}\right|_{\lambda=0} \right) \lambda^k$$

to find $Q_N(\lambda)$ and then use the result of part (a) to find $M_N(s)$. Verify that this moment-generating function agrees with what you would find directly from Eq. (1).

Problem 6.3

Here we shall examine a quantum photodetection model for single-mode direct detection with sub-unity quantum efficiency. Suppose that the sensitive region, $\mathcal{A}$, of a quantum-efficiency-η photodetector is illuminated by a photon-units, positive-frequency quantum field operator $\hat{E}(x, y, t)$ whose only excited, i.e., non-vacuum-state, mode is $\hat{a}e^{-j\omega t}/\sqrt{AT}$ for $0 \le t \le T$ where A is the area of $\mathcal{A}$, as shown in Fig. 2. The output of this detector is a classical random variable N' whose statistics coincide with those of the number operator

$$\hat{N}' \equiv \hat{a}'^{\dagger}\hat{a}' \quad \text{where} \quad \hat{a}' \equiv \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{a}_\eta. \tag{3}$$

In Eq. (3), $\hat{a}_\eta$ is a photon annihilation operator that commutes with $\hat{a}$ and $\hat{a}^\dagger$; $\hat{a}_\eta$ is in its vacuum state $|0\rangle_\eta$.

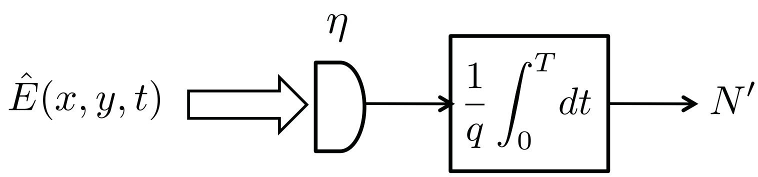

Figure 2: Sub-unity-quantum efficiency photon counter

(a) Find the factorial moments $\{\langle \hat{a}'^{\dagger k}\hat{a}'^k\rangle : k = 1, 2, 3, \ldots\}$ in terms of η and $\{\langle \hat{a}^{\dagger k}\hat{a}^k\rangle : k = 1, 2, 3, \ldots\}$.

(b) Use the result of part (a) to relate $Q_{N'}(\lambda)$ to $Q_N(\lambda)$ from Eq. (2).

(c) Use the result of part (b) to relate $M_{N'}(s)$ to $M_N(s)$ from Eq. (1).

(d) Verify that your answer to part (c) satisfies,

$$M_{N'}(s) = \sum_{n=0}^{\infty} e^{sn} \left[\sum_{k=n}^{\infty} \binom{k}{n} \eta^n (1-\eta)^{k-n} \langle k|\hat{\rho}|k\rangle \right],$$

where

$$\binom{k}{n} \equiv \frac{k!}{n!(k-n)!},$$

is the binomial coefficient. [**Hint:** Interchange the orders of summation over n and k and use the binomial theorem on the resulting inner sum.]

(e) Use the result of part (d) to find $\Pr(N' = n)$ for the quantum-efficiency-η photodetector in terms of $\Pr(N = n)$, the photon counting probability distribution of a unity-quantum-efficiency detector, when the state of the single-mode illumination field is $\hat{\rho}$.

Problem 6.4

Here we shall continue our investigation of quantum linear transformations by treating the single-frequency quantum theory of the degenerate parametric amplifier (DPA), i.e., the Bogoliubov transformation that produces squeezed states. Let $\hat{a}_{\text{IN}}$ be the annihilation operator of the frequency-ω quantum field at the input to the DPA. This operator has the usual commutator bracket with its adjoint, viz., $[\hat{a}_{\text{IN}}, \hat{a}_{\text{IN}}^\dagger] = 1$. The annihilation operator of the frequency-ω output from the DPA is,

$$\hat{a}_{\text{OUT}} \equiv \mu \hat{a}_{\text{IN}} + \nu \hat{a}_{\text{IN}}^\dagger,$$

where μ and ν are complex numbers that satisfy $|\mu|^2 - |\nu|^2 = 1$.

(a) Show that the DPA transformation is commutator preserving, i.e., prove that $[\hat{a}_{\text{OUT}}, \hat{a}_{\text{OUT}}^\dagger] = 1$.

(b) Suppose that the input mode's density operator is $\hat{\rho}_{\text{IN}} = |\alpha_{\text{IN}}\rangle_{\text{IN}\,\text{IN}}\langle\alpha_{\text{IN}}|$, where $|\alpha_{\text{IN}}\rangle_{\text{IN}}$ is a coherent state. Find the Wigner characteristic function,

$$\chi_W^{\rho_{\text{IN}}}(\zeta^*, \zeta) \equiv \text{tr}\left(\hat{\rho}_{\text{IN}} e^{-\zeta^* \hat{a}_{\text{IN}} + \zeta \hat{a}_{\text{IN}}^\dagger}\right),$$

of $\hat{\rho}_{\text{IN}}$.

(c) Find $\chi_W^{\rho_{\text{OUT}}}(\zeta^*, \zeta)$, the Wigner characteristic function of the output mode $\hat{a}_{\text{OUT}}$ by: (1) using the DPA's input-output relation to write the exponential term in the output-mode's characteristic function in terms of the input-mode's annihilation and creation operators, and (2) evaluating the expectation of the resulting exponential term by multiplying by the input-mode density operator and taking the trace.

(d) Suppose that μ and ν are real-valued and positive. Use the result of (c) to find the marginal probability densities for the outcome of the output-mode quadrature measurements,

$$\hat{a}_{\text{OUT}_1} \equiv \frac{\hat{a}_{\text{OUT}} + \hat{a}_{\text{OUT}}^\dagger}{2} \quad \text{and} \quad \hat{a}_{\text{OUT}_2} \equiv \frac{\hat{a}_{\text{OUT}} - \hat{a}_{\text{OUT}}^\dagger}{2j}.$$

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