Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 2 Solutions
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Problem 2.1
Here we shall explore the use of wave plates to perform polarization transformations on a single photon.

(a) It is trivial to argue that the polarization state $\mathbf{i}'$ is identical to the polarization state $\mathbf{i}$. We don't care about the time at which $\text{Re}[\mathbf{i}e^{-j\omega t}]$ or $\text{Re}[\mathbf{i}'e^{-j\omega t}]$ pass particular points in the x-y plane, but only about the full contours they trace out. Thus, because the $z = L$ time evolution is merely the $z = 0$ time evolution delayed by nL/c seconds, the two polarization states are identical. Comparing $\mathbf{i}$ and $\mathbf{i}'$, this implies that two complex-valued unit vectors represent the same state of polarization if one differs from the other by only a scalar phase factor, viz., $\mathbf{i}' = \mathbf{i}e^{j\phi}$. Thus, although we need four real numbers to specify the polarization vector $\mathbf{i}$, we can assume $\alpha_x = |\alpha_x|$ without loss of generality, because the polarization state of the photon depends on the relative phase between α_x and α_y, but *not* on their absolute phases. Furthermore, because $\mathbf{i}$ has unit length, we know that $|\alpha_y| = \sqrt{1 - |\alpha_x|^2}$. Hence, only two real numbers are needed to specify the polarization state of our monochromatic photon.

(b) From Eq. (2) on the problem set, we have that when

$$\mathbf{i} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

is the input to a QWP whose principal axes are aligned with x and y, respectively, the output polarization is

$$\mathbf{i}' = \begin{bmatrix} e^{j\phi_x}/\sqrt{2} \\ e^{j\phi_y}/\sqrt{2} \end{bmatrix} = e^{j\phi_x} \begin{bmatrix} 1/\sqrt{2} \\ e^{-j\pi/2}/\sqrt{2} \end{bmatrix} = e^{j\phi_x} \begin{bmatrix} 1/\sqrt{2} \\ -j/\sqrt{2} \end{bmatrix}$$

The contour traced out by this photon,

$$\text{Re}[\mathbf{i}'e^{-j\omega t}] = \begin{bmatrix} \cos(\omega t - \phi_x)/\sqrt{2} \\ -\sin(\omega t - \phi_x)/\sqrt{2} \end{bmatrix},$$

is a circle, so this is a circularly-polarized photon. Indeed it is a *left*-circularly polarized photon, because the circle that it traces progresses from $+x$ to $-y$.

Now, when this circularly polarized output is the input to *another* QWP whose principal axes are aligned with x and y, respectively, the output polarization—from another application of Eq. (2)—will be

$$\mathbf{i}'' = e^{j\phi_x} \begin{bmatrix} e^{j\phi_x}/\sqrt{2} \\ -je^{j\phi_y}/\sqrt{2} \end{bmatrix} = e^{2j\phi_x} \begin{bmatrix} 1/\sqrt{2} \\ -je^{-j\pi/2}/\sqrt{2} \end{bmatrix} = e^{2j\phi_x} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix},$$

which is $-45°$ linear polarization, because the leading phase factor of $e^{2j\phi_x}$ does not affect the polarization state.

Had we asked you to apply $\mathbf{i}''$ to yet another QWP with principal axes aligned with x and y, respectively, the resulting output polarization would have been right circular. In short, a QWP changes circular into diagonal ($\pm 45°$) polarization, and vice versa.

(c) To solve this HWP problem, we need to rewrite the input polarization state,

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

in the rotated basis corresponding to the fast and slow axes of the HWP, viz.,

$$\vec{i}_{\text{fast}} = \vec{i}_x \cos(\theta) + \vec{i}_y \sin(\theta),$$

and

$$\vec{i}_{\text{slow}} = -\vec{i}_x \sin(\theta) + \vec{i}_y \cos(\theta).$$

We have that the component of $\mathbf{i}$ along $\vec{i}_{\text{fast}}$ is $\cos(\theta)$ and the component along $\vec{i}_{\text{slow}}$ is $-\sin(\theta)$. These components incur phase shifts ϕ_{fast} and ϕ_{slow}, respectively, where $\phi_{\text{slow}} - \phi_{\text{fast}} = \pi$. Thus the output polarization is,

$$\begin{aligned} \mathbf{i}' &= e^{j\phi_{\text{fast}}} \cos(\theta)\vec{i}_{\text{fast}} - e^{j\phi_{\text{slow}}} \sin(\theta)\vec{i}_{\text{slow}} = e^{j\phi_{\text{fast}}}[\cos(\theta)\vec{i}_{\text{fast}} - e^{j\pi} \sin(\theta)\vec{i}_{\text{slow}}] \\ &= e^{j\phi_{\text{fast}}}[\cos(\theta)\vec{i}_{\text{fast}} + \sin(\theta)\vec{i}_{\text{slow}}]. \end{aligned}$$

Converting back to the x-y basis we then find that,

$$\mathbf{i}' = e^{j\phi_{\text{fast}}} \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) \\ 2\sin(\theta)\cos(\theta) \end{bmatrix} = e^{j\phi_{\text{fast}}} \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix},$$

where the second equality follows from standard trig identities. We see from this result that the output is linearly polarized along the direction $\vec{i}_x \cos(2\theta) + \vec{i}_y \sin(2\theta)$, i.e., at twice the angle that the HWP's principal axes made with respect to x and y. Thus, an HWP provides a means for *rotating* linear polarization.

(d) This polarization transformation process is easy to design. We know that an HWP rotates linearly polarized light, and we can see that

$$\mathbf{i}_{\text{HWP}} = \begin{bmatrix} |\alpha_x| \\ |\alpha_y| \end{bmatrix},$$

is linearly polarized. In particular, because our input is x-polarized, we can define $\cos(2\theta) = |\alpha_x|$, $\sin(2\theta) = |\alpha_y|$ and accomplish the desired $\mathbf{i}_{\text{in}}$-to-$\mathbf{i}_{\text{HWP}}$

transformation by arranging the HWP to have its fast and slow axes aligned with the unit vectors

$$\vec{i}_{\text{fast}} = \vec{i}_x \cos(\theta) + \vec{i}_y \sin(\theta),$$

and

$$\vec{i}_{\text{slow}} = -\vec{i}_x \sin(\theta) + \vec{i}_y \cos(\theta).$$

Now to get from $\mathbf{i}_{\text{HWP}}$ to the desired output state $\mathbf{i}_{\text{out}}$ we need only to impose the necessary relative phase between the x and y components of $\mathbf{i}_{\text{HWP}}$. Clearly, this can be done by using another wave plate, whose principal axes are aligned with x and y respectively, and whose propagation phase difference $\phi_x - \phi_y$ satisfies

$$e^{j(\phi_x - \phi_y)} = \frac{\alpha_x \alpha_y^*}{|\alpha_x||\alpha_y|}.$$

Note that we have converted an x-polarized photon into an arbitrary polarization by this procedure. With a little more work, we can show that we can use wave plates to convert an arbitrary *input* polarization into some other arbitrary *output* polarization. First use a wave plate with principal axes along x and y and an appropriate phase difference $\phi_x - \phi_y$ to convert the input polarization to linear. Then rotate that linear polarization to match the $|\alpha_x|$ and $|\alpha_y|$ of the desired output state. Finally, use another wave plate with principal axes aligned with x and y to impart the appropriate phase shift between these $|\alpha_x|$ and $|\alpha_y|$ components.

(e) An arbitrary input polarization

$$\mathbf{i}_{\text{in}} = \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

that is not linear is, in general, an elliptical polarization. Thus, there is a Cartesian coordinate system, (x', y'), in which this input polarization takes the form

$$\mathbf{i}_{\text{in}} = \begin{bmatrix} \alpha'_x \\ \alpha'_y \end{bmatrix},$$

with $\alpha'_y = jk\alpha'_x$, for k a positive constant. If we send this photon through a QWP with its fast axis aligned in the y' direction, we will obtain an output whose polarization vector, in the (x', y') basis, is given by,

$$\mathbf{i}_{\text{QWP}} = \begin{bmatrix} e^{j\phi_{x'}}\alpha'_x \\ e^{j\phi_{y'}}\alpha'_y \end{bmatrix} = \begin{bmatrix} je^{j\phi_{y'}}\alpha'_x \\ je^{j\phi_{y'}}k\alpha'_x \end{bmatrix},$$

which is easily seen to be linearly polarized. An HWP can then be used to rotate $\mathbf{i}_{\text{QWP}}$ so that the photon is linearly polarized in the x direction. Conversely, if we start with x-polarization and want to transform to an arbitrary (x, y)-basis elliptical polarization,

$$\mathbf{i} = \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

which is of the form

$$\mathbf{i} = \begin{bmatrix} \alpha'_x \\ jk\alpha'_x \end{bmatrix}, \quad \text{where } k > 0.,$$

in some (x', y') basis, we proceed as follows. First, we perform an HWP polarization rotation to obtain a linearly-polarized photon with

$$\mathbf{i}_{\text{HWP}} = \begin{bmatrix} |\alpha_x| \\ |\alpha_y| \end{bmatrix},$$

in the (x, y) basis. Then, we employ a QWP, whose fast axis is aligned with x', and we obtain the desired result.

Problem 2.2

Here we shall study the Poincaré sphere, viz., a 3-D real representation for the 2-D polarization state

$$\mathbf{i} = \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

of a $+z$-propagating, frequency-ω photon, i.e., the real-valued 3-vector $\mathbf{r}$ given by,

$$\mathbf{r} \equiv \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2\text{Re}[\alpha_x^*\alpha_y] \\ 2\text{Im}[\alpha_x^*\alpha_y] \\ |\alpha_x|^2 - |\alpha_y|^2 \end{bmatrix}.$$

(a) We have that $|\alpha_x|^2 = 1 - |\alpha_y|^2$. Thus,

$$r_3 = 2|\alpha_x|^2 - 1 = 1 - 2|\alpha_y|^2,$$

whence,

$$|\alpha_x| = \sqrt{(1 + r_3)/2} \quad \text{and} \quad |\alpha_y| = \sqrt{(1 - r_3)/2}.$$

Now, from Problem 2.1(a), we know that we only need the phase difference between α_x and α_y to completely pin down the polarization state. Writing the polar forms,

$$\alpha_x = |\alpha_x|e^{j\theta_x} \quad \text{and} \quad \alpha_y = |\alpha_y|e^{j\theta_y},$$

we see that

$$e^{-j(\theta_x - \theta_y)} = \frac{\alpha_x^*\alpha_y}{|\alpha_x||\alpha_y|} = \frac{r_1 + jr_2}{\sqrt{1 - r_3^2}}.$$

(b) We have that

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 &= 4[\mathrm{Re}(\alpha_x^*\alpha_y)]^2 + 4[\mathrm{Im}(\alpha_x^*\alpha_y)]^2 + (|\alpha_x|^2 - |\alpha_y|^2)^2 \\ &= 4|\alpha_x^*\alpha_y|^2 + |\alpha_x|^4 - 2|\alpha_x|^2|\alpha_y|^2 + |\alpha_y|^4 \\ &= (|\alpha_x|^2 + |\alpha_y|^2)^2 = 1^2 = 1. \end{aligned}$$

Thus a complex-valued unit vector $\mathbf{i}$ maps to a unit-length $\mathbf{r}$ vector. In general, it takes three real numbers to describe a 3-D real-valued vector, but, because $\mathbf{r}$ has unit length, only two real numbers are needed to characterize the polarization of our monochromatic photon in the Poincaré-sphere representation, in agreement with what we found in Problem 2.1(a) for the $\mathbf{i}$ representation.

(c) Here we shall identify the locations of some interesting polarization states on the Poincaré sphere. Linear polarization along the x axis,

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

becomes

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

i.e., the "north pole," if we tranlate (r_1, r_2, r_3) into (x, y, z) the coordinates of $\mathcal{R}^3$. Likewise, linear polarization along the y axis,

$$\mathbf{i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

becomes

$$\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

viz., the "south pole." Right and left circular polarization are then,

$$\mathbf{i} = \begin{bmatrix} 1/\sqrt{2} \\ \pm j/\sqrt{2} \end{bmatrix};$$

and they become,

$$\mathbf{r} = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix},$$

i.e., they lie on the "equator."

Note that x and y polarizations—which are orthogonal, i.e., their complex-unit vectors satisfy $\mathbf{i}_x^\dagger \mathbf{i}_y = 0$—map onto vectors on the Poincaré sphere that are at opposite poles, viz., $\mathbf{r}_x = -\mathbf{r}_y$. Left-circular and right-circular polarizations are also orthogonal, and they two map into vectors on the Poincaré sphere that satisfy $\mathbf{r}_{\text{left}} = -\mathbf{r}_{\text{right}}$. These occurrences are *not* accidental: any two orthogonal polarizations—$\mathbf{i}$ and $\mathbf{i}'$ satisfying $\mathbf{i}^\dagger\mathbf{i}' = 0$—map into vectors on the Poincaré sphere that satisfy $\mathbf{r} = -\mathbf{r}'$.

(d) Let

$$\mathbf{i} \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} \quad \text{and} \quad \mathbf{r} \equiv \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2\text{Re}[\alpha_x^*\alpha_y] \\ 2\text{Im}[\alpha_x^*\alpha_y] \\ |\alpha_x|^2 - |\alpha_y|^2 \end{bmatrix}$$

be equivalent representations of the polarization state of a monochromatic photon, and let

$$\mathbf{i}' \equiv \begin{bmatrix} \alpha'_x \\ \alpha'_y \end{bmatrix} \quad \text{and} \quad \mathbf{r}' \equiv \begin{bmatrix} r'_1 \\ r'_2 \\ r'_3 \end{bmatrix} = \begin{bmatrix} 2\text{Re}[\alpha'^*_x\alpha'_y] \\ 2\text{Im}[\alpha'^*_x\alpha'_y] \\ |\alpha'_x|^2 - |\alpha'_y|^2 \end{bmatrix}$$

be another pair of equivalent polarizations. We then have that

$$|\mathbf{i}'^\dagger\mathbf{i}|^2 = |\alpha'^*_x\alpha_x + \alpha'^*_y\alpha_y|^2 = |\alpha'_x|^2|\alpha_x|^2 + |\alpha'_y|^2|\alpha_y|^2 + 2\text{Re}[\alpha'^*_x\alpha'_y\alpha_x\alpha_y^*],$$

and

$$\begin{aligned} \mathbf{r}'^T\mathbf{r} &= 4\text{Re}[\alpha'^*_x\alpha'_y]\text{Re}[\alpha_x^*\alpha_y] + 4\text{Im}[\alpha'^*_x\alpha'_y]\text{Im}[\alpha_x^*\alpha_y] \\ &+ (|\alpha'_x|^2 - |\alpha'_y|^2)(|\alpha_x|^2 - |\alpha_y|^2) \\ &= 4\text{Re}[\alpha'^*_x\alpha'_y\alpha_x\alpha_y^*] + (|\alpha'_x|^2 - |\alpha'_y|^2)(|\alpha_x|^2 - |\alpha_y|^2) \\ &+ (|\alpha'_x|^2 + |\alpha'_y|^2)(|\alpha_x|^2 + |\alpha_y|^2) - 1 \\ &= 4\text{Re}[\alpha'^*_x\alpha'_y\alpha_x\alpha_y^*] + 2(|\alpha'_x|^2|\alpha_x|^2 + |\alpha'_y|^2|\alpha_y|^2) - 1, \end{aligned}$$

from which it is trivial to verify that

$$|\mathbf{i}'^\dagger\mathbf{i}|^2 = \frac{1 + \mathbf{r}'^T\mathbf{r}}{2}.$$

The equivalence we have just proven will be of use later, when we study polarization entanglement. Also note that $\mathbf{i}'^\dagger\mathbf{i} = 0$ implies that $\mathbf{r}'^T\mathbf{r} = -1$, and vice versa, as seen for specific examples in part (c). In other words, orthogonal polarizations have antipodal Poincaré-sphere vectors, viz., $\mathbf{r}' = -\mathbf{r}$ when $\mathbf{r}'$ and $\mathbf{r}$ correspond to $\mathbf{i}'$ and $\mathbf{i}$ satisfying $\mathbf{i}'^\dagger\mathbf{i} = 0$.

Problem 2.3

Here we shall introduce the notion of matrix elements for a linear operator on the Hilbert space $\mathcal{H}$.

(a) Using the completeness relation for the $\{|\phi_n\rangle\}$, we have that

$$\begin{aligned}\hat{A} &= \hat{I}\hat{A}\hat{I} = \left(\sum_{m=1}^{\infty}|\phi_m\rangle\langle\phi_m|\right)\hat{A}\left(\sum_{n=1}^{\infty}|\phi_n\rangle\langle\phi_n|\right)\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\phi_m\rangle(\langle\phi_m|\hat{A}|\phi_n\rangle)\langle\phi_n| = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\langle\phi_m|\hat{A}|\phi_n\rangle)|\phi_m\rangle\langle\phi_n|,\end{aligned}$$

where the last equality uses the fact that the matrix elements are numbers.

(b) Using the result from (a) we have that

$$|y\rangle = \hat{A}|x\rangle = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\langle\phi_m|\hat{A}|\phi_n\rangle)|\phi_m\rangle\langle\phi_n|x\rangle = \sum_{m=1}^{\infty} y_m|\phi_m\rangle,$$

where

$$y_m \equiv \sum_{m=1}^{\infty}(\langle\phi_m|\hat{A}|\phi_n\rangle)x_n, \quad \text{with } x_n \equiv \langle\phi_n|x\rangle.$$

(c) If $\hat{A}$ is an observable and the $\{|\phi_n\rangle\}$ are its CON eigenkets, then the matrix elements of $\hat{A}$ satisfy,

$$\langle\phi_m|\hat{A}|\phi_n\rangle = \mu_n\delta_{nm},$$

where the $\{\mu_n\}$ are the eigenvalues associated with the $\{|\phi_n\rangle\}$ eigenkets. To prove that this is so, we merely introduce the eigenvalue/eigenket relation,

$$\hat{A}|\phi_n\rangle = \mu_n|\phi_n\rangle, \quad \text{for } 1 \le n < \infty,$$

and employ the orthonormality of the eigenkets to obtain,

$$\langle\phi_m|\hat{A}|\phi_n\rangle = \mu_n\langle\phi_m|\phi_n\rangle = \mu_n\delta_{nm}.$$

It then follows from (a) that,

$$\hat{A} = \sum_{n=1}^{\infty}\mu_n|\phi_n\rangle\langle\phi_n|,$$

and from (b) that,

$$|y\rangle = \sum_{n=1}^{\infty}\mu_n x_n|\phi_n\rangle.$$

Problem 2.4

Here we derive the stationary-state property of the Hamiltonian's eigenkets.

(a) The time-evolution operator obeys the Schrödinger equation

$$j\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}\hat{U}(t,t_0), \quad \text{for } t \geq t_0,$$

with the initial condition $\hat{U}(t_0,t_0) = \hat{I}$. This operator-valued differential equation can be converted into an infinite set of coupled classical differential equations, by taking the $\{|h_n\rangle\}$ matrix elements of both sides. The result is,

$$j\hbar\frac{\partial}{\partial t}\langle h_m|\hat{U}(t,t_0)|h_n\rangle = \langle h_m|\hat{H}\hat{U}(t,t_0)|h_n\rangle, \quad \text{for } t \geq t_0,\ 0 \leq n,m < \infty,$$

with the initial conditions

$$\langle h_m|\hat{U}(t_0,t_0)|h_n\rangle = \langle h_m|\hat{I}|h_n\rangle = \delta_{nm}.$$

Now, using the fact that the $\{|h_n\rangle\}$ are the eigenkets of $\hat{H}$ with associated eigenvalues $\{h_n\}$, we get

$$j\hbar\frac{\partial}{\partial t}\langle h_m|\hat{U}(t,t_0)|h_n\rangle = h_m\langle h_m|\hat{U}(t,t_0)|h_n\rangle.$$

For $m \neq n$, we need the solution to this homogeneous linear differential equation subject to the initial condition $\langle h_m|\hat{U}(t_0,t_0)|h_n\rangle = 0$. The answer, of course, is

$$\langle h_m|\hat{U}(t,t_0)|h_n\rangle = 0, \quad \text{for } t \geq t_0 \text{ when } m \neq n.$$

For $m = n$, we need to solve,

$$j\hbar\frac{\partial}{\partial t}\langle h_n|\hat{U}(t,t_0)|h_n\rangle = h_n\langle h_n|\hat{U}(t,t_0)|h_n\rangle, \quad \text{for } t \geq t_0,$$

subject to the initial condition,

$$\langle h_n|\hat{U}(t_0,t_0)|h_n\rangle = 1,$$

The solution is easily found:

$$\langle h_n|\hat{U}(t,t_0)|h_n\rangle = \exp[-jh_n(t-t_0)/\hbar], \quad \text{for } 1 \leq n < \infty.$$

The matrix elements of an operator in a CON basis determine that operator, as shown in Problem 2.3 (a). For the case at hand now, we have that

$$\hat{U}(t,t_0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle h_m|\hat{U}(t,t_0)|h_n\rangle|h_m\rangle\langle h_n| = \sum_{n=0}^{\infty}\exp[-jh_n(t-t_0)/\hbar]|h_n\rangle\langle h_n|,$$

QED.

(b) We can derive this commutator result from the Schrödinger equation or from the matrix elements we've just found in (a). Let's take the latter approach here. We have that

$$\begin{aligned}\langle h_m|\left[\hat{U}(t,t_0),\hat{H}\right]|h_n\rangle &= \langle h_m|\left[\hat{U}(t,t_0)\hat{H}-\hat{H}\hat{U}(t,t_0)\right]|h_n\rangle \\ &= \langle h_m|\hat{U}(t,t_0)\hat{H}|h_n\rangle - \langle h_m|\hat{H}\hat{U}(t,t_0)|h_n\rangle \\ &= h_n\langle h_m|\hat{U}(t,t_0)|h_n\rangle - h_m\langle h_m|\hat{U}(t,t_0)|h_n\rangle,\end{aligned}$$

where we have used the fact the $\hat{H}$ is Hermitian, and hence its eigenvalues are real. The matrix elements from (a) now give us our first desired result,

$$\left[\hat{U}(t,t_0),\hat{H}\right]=0.$$

The derivation of

$$\left[\hat{U}^\dagger(t,t_0),\hat{H}\right]=0,$$

is essentially the same:

$$\begin{aligned}\langle h_m|\left[\hat{U}^\dagger(t,t_0),\hat{H}\right]|h_n\rangle &= \langle h_m|\left[\hat{U}^\dagger(t,t_0)\hat{H}-\hat{H}\hat{U}^\dagger(t,t_0)\right]|h_n\rangle \\ &= \langle h_m|\hat{U}^\dagger(t,t_0)\hat{H}|h_n\rangle - \langle h_m|\hat{H}\hat{U}^\dagger(t,t_0)|h_n\rangle \\ &= h_n\langle h_m|\hat{U}^\dagger(t,t_0)|h_n\rangle - h_m\langle h_m|\hat{U}^\dagger(t,t_0)|h_n\rangle.\end{aligned}$$

Using $\langle h_m|\hat{U}^\dagger(t,t_0)|h_n\rangle = \langle h_n|\hat{U}(t,t_0)|h_m\rangle^*$, in conjunction with the results from (a), completes the proof.

(c) First, expand $|\psi(t_0)\rangle$ in the $\{|h_n\rangle\}$ basis:

$$|\psi(t_0)\rangle = \sum_{n=0}^{\infty}\psi_n(t_0)|h_n\rangle, \quad \text{where } \psi_n(t_0)\equiv\langle h_n|\psi(t_0)\rangle.$$

Next, use the results of (a) and Problem 2.3 to get,

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle = \sum_{n=0}^{\infty}\exp[-jh_n(t-t_0)/\hbar]\psi_n(t_0)|h_n\rangle.$$

This result holds for an arbitrary initial state. We are given that $|\psi(t_0)\rangle = |h_1\rangle$. Thus, $\psi_n(t_0)=\delta_{n1}$, and so

$$|\psi(t)\rangle = \exp[-jh_1(t-t_0)/\hbar]|h_1\rangle = \exp[-jh_1(t-t_0)/\hbar]|\psi(t_0)\rangle.$$

(d) We know that the outcome of the $\hat{O}$ measurement at time t will be one of the eigenvalues, $\{o_k\}$, and that,

$$\Pr(\hat{O}\text{-measurement outcome} = o_k) = |\langle o_k|\psi(t)\rangle|^2, \quad \text{for } k = 1, 2, 3, \ldots$$

Using the result of (c), we see that this probability distribution is the same for all $t \geq t_0$, when $|\psi(t_0)\rangle = |h_1\rangle$. Because $|h_1\rangle$ is an arbitrary eigenket of the Hamiltonian, this means that these eigenkets are stationary states, i.e., when any observable is measured on a system in the eigenket of a (time-independent) Hamiltonian, the resulting measurement statistics are independent of the time at which that measurement was made.

Problem 2.5

Here we shall derive the time-frequency uncertainty principle of classical signal analysis. Essentially the same derivation can lead to the Heisenberg uncertainty principle for position and momentum by means of wave function (rather than Dirac-notation) quantum mechanics.

(a) Because $|x(t)|^2 \geq 0$ for all t, and because $|X(f)|^2 \geq 0$ for all f, it is clear that $p(t) \geq 0$ for all t and $P(f) \geq 0$ for all f. We have that

$$\int_{-\infty}^{\infty} dt\, p(t) = \int_{-\infty}^{\infty} dt\, K|x(t)|^2 = K \int_{-\infty}^{\infty} dt\, |x(t)|^2 = 1,$$

where $K \equiv 1/\int_{-\infty}^{\infty} dt\, |x(t)|^2$. Likewise,

$$\int_{-\infty}^{\infty} df\, P(f) = \int_{-\infty}^{\infty} df\, K'\, |X(f)|^2 = K' \int_{-\infty}^{\infty} df\, |X(f)|^2 = 1,$$

where $K' \equiv 1/\int_{-\infty}^{\infty} df\, |X(f)|^2$. Thus, both $p(t)$ and $P(f)$ are properly normalized, non-negative functions, hence they can be thought of as probability densities. Note that Parseval's theorem tells us that $K = K'$ in the above derivation.

(b) The inverse tranform integral that relates $X(f)$ back to $x(t)$ is

$$x(t) = \int_{-\infty}^{\infty} df\, X(f) e^{j2\pi f t}.$$

Differentiating both sides of this equation with respect to t, and bringing the derivative inside the f-integral on the right-hand side gives the desired result:

$$\frac{dx(t)}{dt} = \int_{-\infty}^{\infty} df\, j2\pi f X(f) e^{j2\pi f t}.$$

Next, using this result and Parseval's theorem, we have that

$$\begin{aligned}
TW &= \frac{\sqrt{\int_{-\infty}^{\infty} dt\, t^2 |x(t)|^2 \int_{-\infty}^{\infty} df\, f^2 |X(f)|^2}}{\int_{-\infty}^{\infty} dt\, |x(t)|^2} \\
&= \frac{1}{2\pi} \frac{\sqrt{\int_{-\infty}^{\infty} dt\, t^2 |x(t)|^2 \int_{-\infty}^{\infty} dt\, \left| \frac{dx(t)}{dt} \right|^2}}{\int_{-\infty}^{\infty} dt\, |x(t)|^2}
\end{aligned}$$

Applying the Schwarz inequality then yields,

$$TW \geq \frac{1}{2\pi} \frac{\left| \int_{-\infty}^{\infty} dt\, t x^*(t) \frac{dx(t)}{dt} \right|}{\int_{-\infty}^{\infty} dt\, |x(t)|^2},$$

with equality if and only if $\frac{dx(t)}{dt} = Ctx(t)$, for $-\infty < t < \infty$, with C a complex number.

(c) Because $|z| \geq |\text{Re}(z)|$ for any complex number z, we can loosen the bound in (b) to the following:

$$TW \geq \frac{1}{2\pi} \frac{\left| \text{Re}\left(\int_{-\infty}^{\infty} dt\, t x^*(t) \frac{dx(t)}{dt} \right) \right|}{\int_{-\infty}^{\infty} dt\, |x(t)|^2},$$

with equality if and only if $x^*(t)\frac{dx(t)}{dt}$ is real valued. Now, expanding the real part in the numerator we have that

$$\begin{aligned}
\text{Re}\left(\int_{-\infty}^{\infty} dt\, t x^*(t) \frac{dx(t)}{dt} \right) &= \frac{1}{2} \left[\int_{-\infty}^{\infty} dt\, \left(t x^*(t) \frac{dx(t)}{dt} + t x(t) \frac{dx^*(t)}{dt} \right) \right] \\
&= \frac{1}{2} \int_{-\infty}^{\infty} dt\, t \frac{d(|x(t)|^2)}{dt} \\
&= \frac{1}{2} \left(t|x(t)|^2 \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dt\, |x(t)|^2 \right) = \frac{1}{2}(0 - 1) = -\frac{1}{2},
\end{aligned}$$

where the second equality uses the chain rule for differentiation, the third equality follows via integration by parts, and the last equality relies on the fact that

$|x(t)|^2$ integrates to unity on $-\infty < t < \infty$. Plugging this result back into our last TW bound completes the proof that $TW \geq 1/4\pi$.

(d) We have already stated that equality occurs in (b) if and only if $\dfrac{dx(t)}{dt} = Ctx(t)$ for C a complex number. Rearranging this equality condion to read

$$\frac{d\ln[x(t)]}{dt} = \frac{1}{x(t)}\frac{dx(t)}{dt} = Ct,$$

leads to the solution

$$\ln[x(t)] = Ct^2/2 + D,$$

where D is another complex number (constant of integration). Exponentiating now yields what we wanted to show: $x(t) = A\exp(at^2)$, where A and a are complex numbers, is a time function that will satisfy the bound in (b) with equality IF $\text{Re}(a) < 0$, so that $\int_{-\infty}^{\infty} dt\,|x(t)|^2 < \infty$.

If we assume that $x(t)$ is of this form, then to satisfy $TW = 1/4\pi$ we need only impose the additional constraint that $x^*(t)\dfrac{dx(t)}{dt}$ be real valued. Substituting in the form we have for $x(t)$ shows that this latter condition is equivalent to requiring that $2|A|^2 at\exp[2\text{Re}(a)t^2]$ be real valued. This only happens when a is real.

The putative $x(t)$ and $X(f)$ Fourier transform pair can be verified by using the characteristic function for the Gaussian probability density function. In particular, we know that

$$\int_{-\infty}^{\infty} dt\,\frac{\exp(-t^2/4t_0^2)}{\sqrt{4\pi t_0^2}}e^{-j2\pi ft} = \exp(-4\pi^2 f^2 t_0^2),$$

which leads to the desired result for $X(f)$ after we multiply both sides of this equation by $\sqrt{4\pi t_0^2}/(2\pi t_0^2)^{1/4} = (8\pi t_0^2)^{1/4}$. Next, because

$$|x(t)|^2 = \frac{\exp(-t^2/2t_0^2)}{\sqrt{2\pi t_0^2}},$$

is a Gaussian probability density with mean zero and variance t_0^2, we obtain $T = t_0$. Similarly, we see that

$$|X(f)|^2 = \sqrt{8\pi t_0^2}\exp(-8\pi^2 f^2 t_0^2),$$

is a Gaussian probability density with mean zero and variance $1/16\pi^2 t_0^2$. Thus we have that $W = 1/4\pi t_0$, and $TW = 1/4\pi$.

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