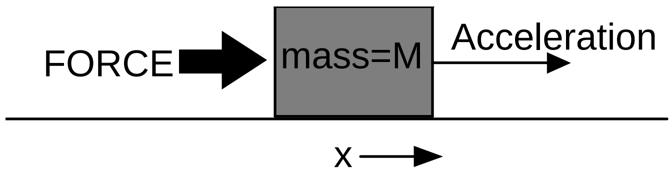
Lecture 7: Lumped Elements

I. What is a lumped element?

Lumped elements are physical structures that act and move as a unit when subjected to controlled forces. Imagine a two-dimensional block of lead on a one-dimensional frictionless surface.

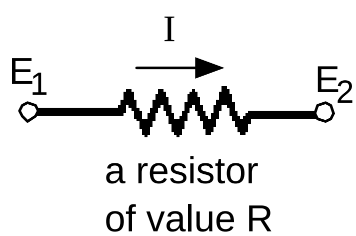


When a force is imposed on the block, the block moves as a unit in a direction described by the difference in force acting on its two surfaces, or analytically:

$$\frac{dV}{dt} = \frac{Net\ Force}{Mass} \qquad (5.1)$$

The key features is that a gradient of a physical parameter produces a uniform physical response throughout the lump.

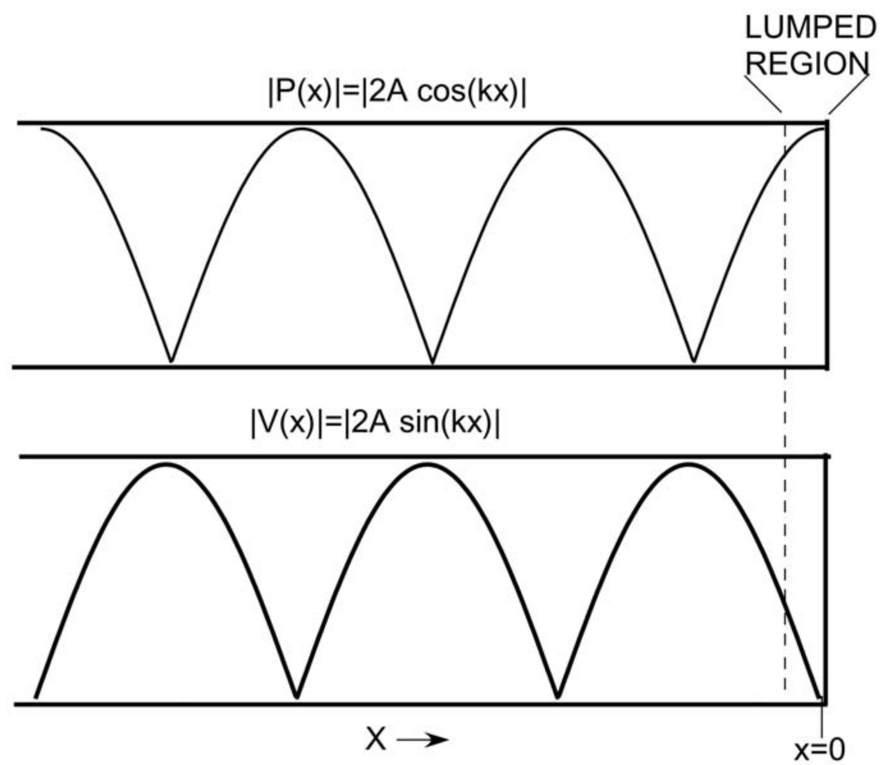
Another example of a lumped element is an electrical resistor where a difference in the Voltage (*E*) across the resistive element produces a current (*I*) that is uniform throughout the resistor:



where: $$I = (E_1 - E_2)/R \qquad (5.2)$$

II. Lumped Acoustic Elements

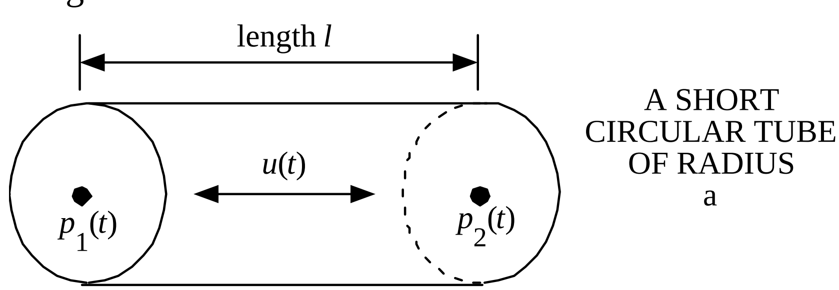
A. Elements: A lumped element is a representation of a structure by one or two physical quantities that are homogenous or varying linearly throughout the structure



Standing Waves in $\underline{P}$ and $\underline{V}$ in a long tube with a rigid termination at *x*=0.

The spatial variation in the sound pressure magnitude and phase $\underline{P}(x)$ is defined by a cosine function. The spatial variation in particle velocity magnitude and phase $\underline{V}(x)$ is defined by a sine function. The region where the tube can act as a lumped element is the region where the pressure amplitude is nearly constant and the 'volume velocity' (v x tube cross-section) varies linear with *x*.

B. *An example of a lumped acoustic element* is a short open tube of moderate diameter, where length *l* and radius *a* are *<0.1* λ.



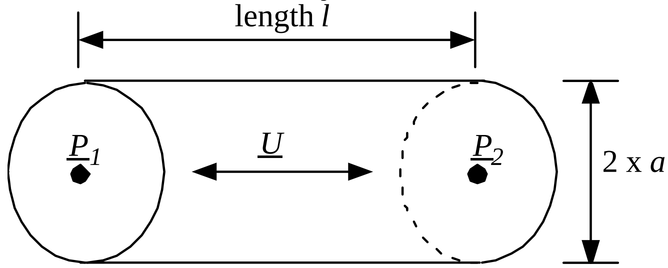
Under these circumstances particle velocity $\underline{V}$ and the sound pressures are simply related by:

$$\frac{d\underline{V}}{dt} = \frac{(\underline{P}_1 - \underline{P}_2)}{\rho_0 l} \qquad (5.3)$$

where Eqn. 5.3 is the specific acoustic equivalent of Eqn. 5.1. (Hint: you can describe the forces acting on the lump by multiplying the pressures by the cross-sectional area of the tube πa^2.

C. Volume Velocity and Acoustic Impedance

In discussing lumped acoustic elements, it is convenient to think about velocity in terms of a new variable Volume Velocity $\underline{U}$ where in the case of the tube above,



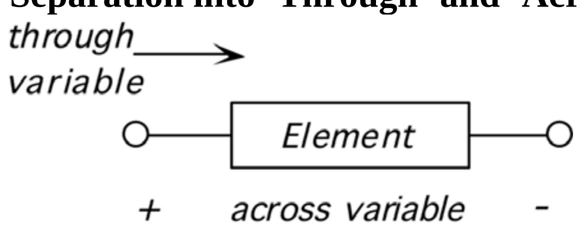
the volume velocity is defined by the product of the particle velocity and the cross-sectional area of the tube, i.e. $\underline{U} = \pi a^2 \underline{V} = S\underline{V}$.

The relationship between volume velocity and the pressure difference in the open tube above can be obtained by multiplying both sides of Eqn, 5.3 by $S = \pi a^2$, i.e.

$$S\frac{d\underline{V}}{dt} = \frac{(\underline{P}_1 - \underline{P}_2)}{\rho_0 l}S$$
$$\frac{d\underline{U}}{dt} = \frac{(\underline{P}_1 - \underline{P}_2)}{\rho_0 l}S = \frac{(\underline{P}_1 - \underline{P}_2)}{\rho_0 Sl}S^2, where\ Sl = Tube\ Volume. \qquad (5.4)$$

The Acoustic Impedance of the tube is : $\dfrac{\underline{P}_1 - \underline{P}_2}{\underline{U}}$

III. Separation into 'Through' and 'Across' Variables



, where $power(t) = through(t)\ across(t)$

	'Across' variable	'Through' variable
Electrics	voltage *e(t)*	current *i(t)*
Mechanics: Impedance analogy	force *f(t)*	velocity *v(t)*
Mechanics: Mobility analogy	velocity *v(t)*	force *f(t)*
Acoustics: Impedance	sound pressure *p(t)*	volume velocity *u(t)*

analogy		
Acoustics: Mobility analogy	volume velocity $u(t)$	sound pressure $p(t)$

In all of the above analogies, $power(t) = through(t)\ across(t)$ has units of watts.

IV. Two Terminal Elements

A. Electrical Elements

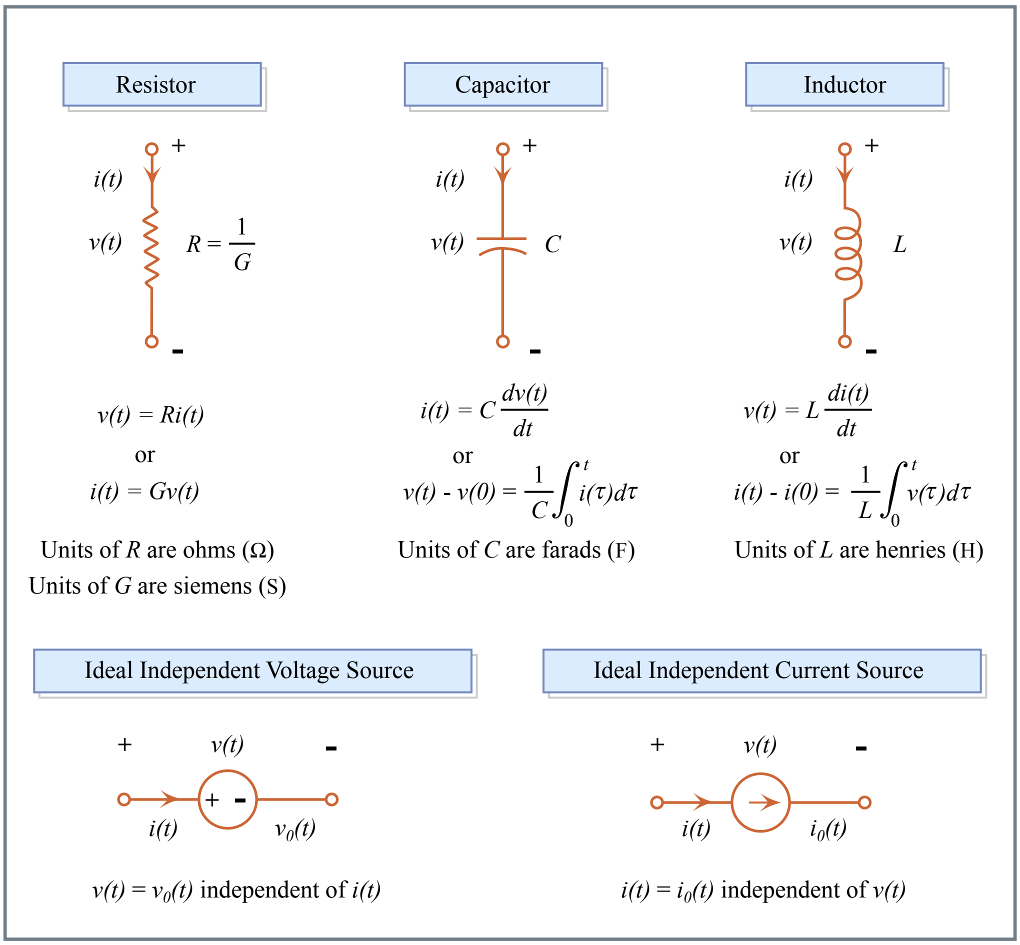


Figure 5.1 Simple linear <u>2-terminal</u> lumped electrical elements and their constitutive relations. The orientation of the arrow and the +/- signs identifies the positive reference direction for each element. In this figure the variable *i* is current and *v* is voltage. (From Siebert "Circuits, Signals and System, 1986).

Note that R, C and L are the coefficients of the 0th and 1st order differential equations that relate $v(t)$ (or $e(t)$) to $i(t)$.

B. *Analogous Elements*

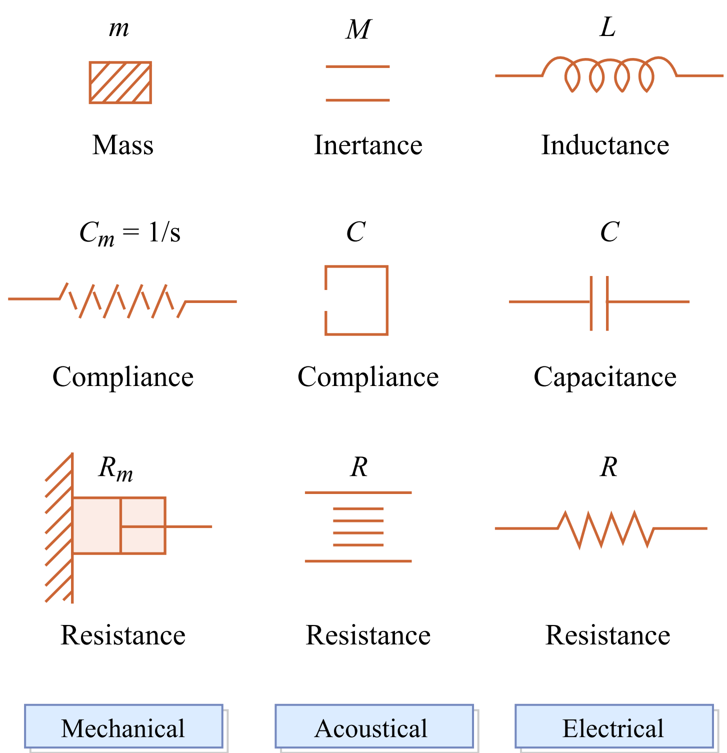


Fig. 10.3. *Acoustic, electrical and mechanical analogues.*

Figure 5.2
Electric elements and their mechanical and acoustic counter-parts in the "Impedance analogy" From Kinsler, Frey, Coppens, & Sanders, *Fundamentals of Acoustics* , 3rd Ed. (1982)

C. *Analogous Constitutive Relationships*

	Mechanical *V vs F*	Electrical *I vs E*	Acoustical *U vs P*
Spring Capacitor Compliance	Spring $v(t) = C_M \dfrac{df(t)}{dt}$	Capacitor $i(t) = C_E \dfrac{de(t)}{dt}$	Compliance $u(t) = C_A \dfrac{dp(t)}{dt}$
Damper Resistor Resistor	Damper $v(t) = \dfrac{1}{R_M} f(t)$	Resistor $i(t) = \dfrac{1}{R_E} e(t)$	Resistor $u(t) = \dfrac{1}{R_A} p(t)$
Mass Inductor Inertance	Mass $v(t) = \dfrac{1}{L_M} \int f(t)dt$	Inductor $i(t) = \dfrac{1}{L_E} \int e(t)dt$	Inertance $u(t) = \dfrac{1}{L_A} \int p(t)dt$

In the Sinusoidal Steady State:

	Mechanical	Electrical	Acoustical

	V vs F	*I vs E*	*U vs P*
Spring Capacitor Compliance	Spring $\underline{V}(\omega) = j\omega C_M \underline{F}(\omega)$	Capacitor $\underline{I}(\omega) = j\omega C_E \underline{E}(\omega)$	Compliance $\underline{U}(\omega) = j\omega C_A \underline{P}(\omega)$
Damper Resistor Resistor	Damper $\underline{V}(\omega) = \frac{1}{R_M} \underline{F}(\omega)$	Resistor $\underline{I}(\omega) = \frac{1}{R_E} \underline{E}(\omega)$	Resistor $\underline{U}(\omega) = \frac{1}{R_A} \underline{P}(\omega)$
Mass Inductor Inertance	Mass $\underline{V}(\omega) = \frac{1}{j\omega L_M} \underline{F}(\omega)$	Inductor $\underline{I}(\omega) = \frac{1}{j\omega L_E} \underline{E}(\omega)$	Inertance $\underline{U}(\omega) = \frac{1}{j\omega L_A} \underline{P}(\omega)$

$$p(t) = \text{Real}\left\{\underline{P}e^{j\omega t}\right\} = |\underline{P}|\cos(\omega t + \angle\underline{P})$$

$$\frac{dp(t)}{dt} = \text{Real}\left\{j\omega\underline{P}e^{j\omega t}\right\} = -\omega|\underline{P}|\sin(\omega t + \angle\underline{P})$$

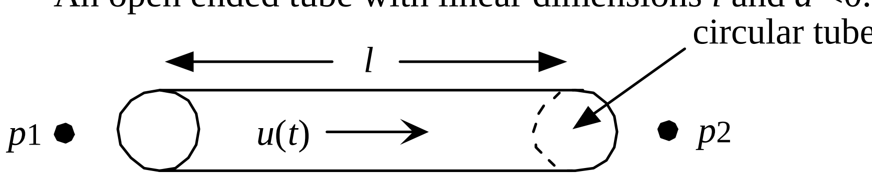
$$= \omega|\underline{P}|\cos(\omega t + \angle\underline{P} + \pi/2)$$

V. Acoustic Element Values and Physics

Element constraints result from physical process and element values are determined by physical properties including the dimensions of structures, e.g. the electrical resistance of a resistor depend on the dimensions and the resistivity of the material from which it’s constructed.

A. Acoustic mass: *units of kg/m⁴*

An open ended tube with linear dimensions l and a <0.1 λ *and* $S=\pi a^2$



$$p(t) = p1(t) - p2(t)$$

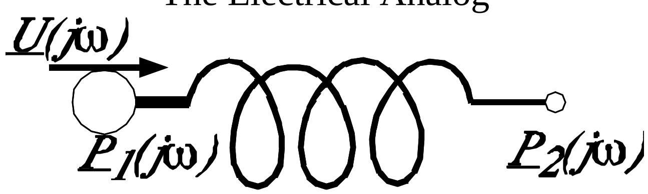
$$p(t) = L_A \frac{du(t)}{dt}$$

assumes only inertial forces

$$L_A = \frac{\rho_o l}{S} = \frac{\rho_o Volume}{S^2}$$

ρ_o = equilibrium mass density of medium

The Electrical Analog



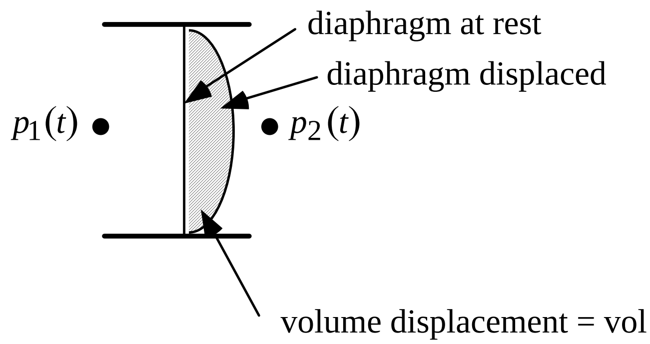
$$\underline{P_1}-\underline{P_2}=\underline{U}\, j\omega L_A\ .$$

Note that the acoustic mass is equivalent to the mass of the air in the enclosed element divided by the square of the cross-sectional area of the element. Also since some small volume of the medium on either end of the tube is also entrained with the media inside the tube, the “acoustic” length is usually somewhat larger than the physical length of the tube. For a single open end, the difference between the physical length and the acoustic length is $\Delta l \approx 0.8a$. This difference is called the *end correction.*

B. Acoustic Compliance: *units of m^3/Pa*

Volume displaced per unit pressure difference (2 examples, both of which assume resistance and inertia are negligible).

1. A Diaphragm of $a < 0.1\,\lambda$



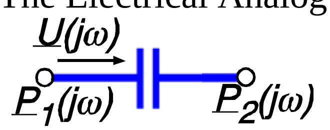
$$\text{vol} = \left(p_1(t) - p_2(t)\right)C_A$$

$$= p(t)C_A$$

$$u(t) = \frac{d(\text{vol})}{dt}$$

$$u(t) = C_A \frac{dp(t)}{dt}$$

The Electrical Analog



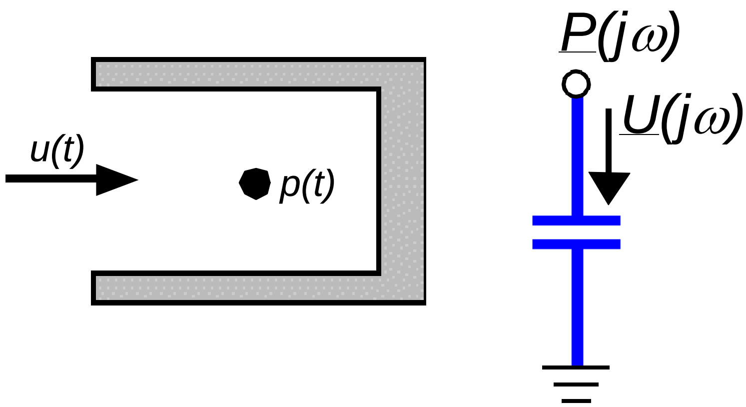
$$\underline{U} = j\omega C_A(\underline{P}_1 - \underline{P}_2)$$

For a round, flat, "simply mounted" plate

$$C_A = \frac{\pi a^6 (7+v)(1-v)}{16Et^3},$$

where: a is the radius of the plate, $v = 0.3$ is Poisson's ratio, E is the elastic constant (Young's modulus) of the material, and t is the thickness (Roark and Young, 1975, p. 362-3, Case 10a).

2. Enclosed volume of air with linear dimensions <0.1 λ



Another structure that may be well approximated by an acoustic compliance.

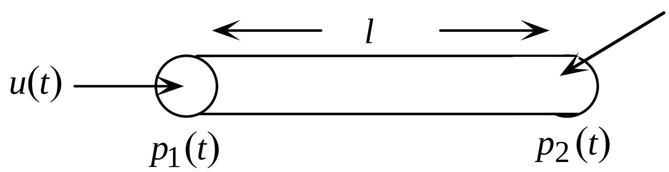
$$C = \frac{\text{Volume}}{\text{Adiabatic Bulk modulus}}$$

$$\underline{U} = j\omega C_A \underline{P}$$

The variations in sound pressure within an enclosed air volume generally occur about the steady-state atmospheric pressure, the ground potential in acoustics. Therefore, one terminal of an electrical-analog of a volume-determined acoustic compliance should always be grounded.

C. Acoustic Resistance*: units of Acoustic Ohms (Pa-s/m³)*

1. A narrow tube or radius *a* << 0.001 λ



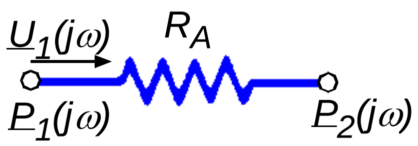
$$p(t) = p_1(t) - p_2(t) = R_A u(t) \leftarrow$$ ***assumption; only viscous forces***

$$R_A = \frac{8\eta l}{\pi a^4} \Rightarrow \frac{p_1(t) - p_2(t)}{u(t)}$$

η = viscosity of medium

Because of the viscous forces, relative motions of fluid at one radial position with respect to an adjacent position exerts a force opposing the motion that is proportional to the spatial derivative of the velocity and the fluid's coefficient of shear viscosity η.

The action of these forces results in a proportionality of pressure difference and volume velocity that is analogous to an electric resistance. In the sinusoidal steady state, then:



, where $\underline{P_1}-\underline{P_2}=\underline{U_1}\,R_A$.

The consequence of the viscosity is that the velocity at the stationary walls is zero, and is maximum in the center of the tube (see Fig. 5.3). The viscous forces produce energy loss near the walls where the velocity changes with position.

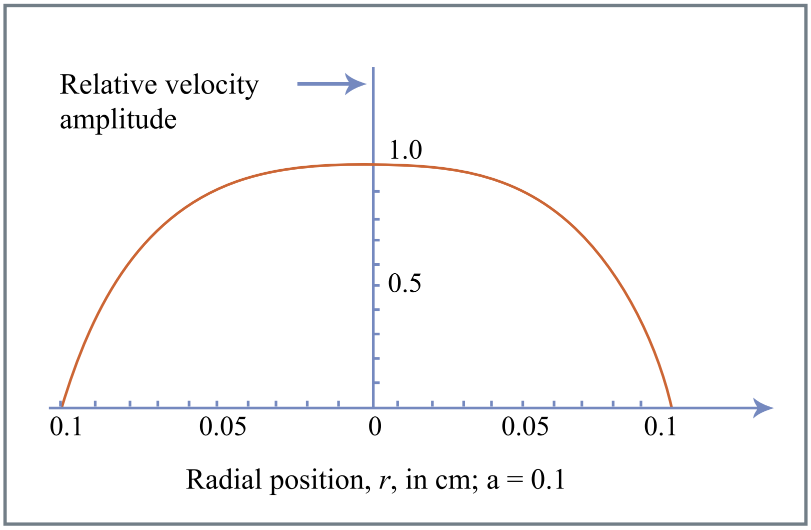


Fig. 5.3 Relative Particle velocity amplitude as a function of radial position in a small pipe of radius *a* = 0.1 cm, at frequency f = 200 Hz.
After Kinsler and Frey, 1950; p. 238

The velocity profile in Figure 5.3 varies as $v(r)=1-e^{\left(\frac{-(0.1-r)}{\delta}\right)}$, where the “space constant” $\delta=\left[\eta/(\rho_0\omega)\right]^{1/2}$, with η, the coefficient of shear viscosity = $1.86\text{x}10^{-5}$ N-s-m^{-2} for air at STP, ρ_0, density of air = 1.2 kg-m^3, and ω, radian frequency = 2πf.

At 200 Hz δ=1.1x10-4 m = 0.011 cm (Figure 5.3)
At 20 Hz δ=3.5x10-5 m = 0.035 cm

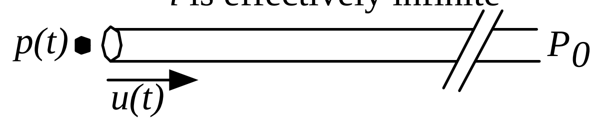
The effect of the viscous forces is insignificant when the radius of the tube is an order of magnitude or more larger than the space constant and therefore we can ignore viscosity for short tubes of moderate to large radius, $0.01\lambda < a < 0.1\lambda$.

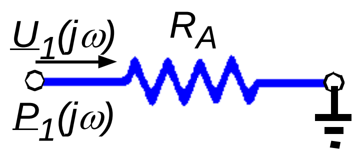
2. An infinitely long tube

The action of an acoustic resistor is to absorb sound power. The viscous forces within a narrow tube convert the sound power into heat that dissipates away. A second type of acoustic resistance can be constructed from a long tube of moderate cross-sectional dimensions ($0.01\,\lambda < a < 0.2\,\lambda$). Such a construction can conduct sound power away from a system and can be treated as an acoustic resistance where:

$$R = \frac{\rho_0 c}{\pi a^2}.$$

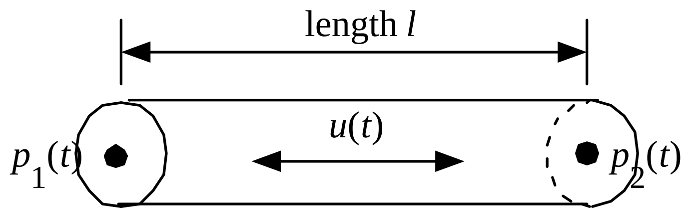
There is a catch, however, in that this lumped element always has one end coupled to ground and therefore can only be used to either terminate acoustic circuits or be placed in parallel with other elements. There are ways of dealing with long tubes as a collection of series and parallel elements that have already been discussed in Lecture 2.





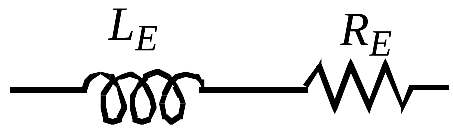
D. Two Mixed mass-resistance acoustic loads

1. A tube of intermediate radius (neither wide nor narrow) has an impedance determined by the combination of an acoustic mass or inertance (associated with accelerating the fluid mass within the tube) and a resistance (associated with overcoming viscous drag at the stationary walls of the tube). Since the pressure drop across the resistance and the mass elements add, we think of these as an R and L in series.



An intermediate tube

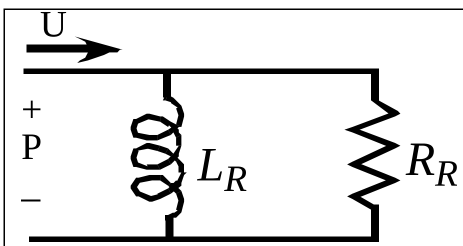
$$\Delta \underline{P} = \underline{P}_2 - \underline{P}_1 = \underline{U}\left(j\omega\rho_o l/S + R\right)$$



where S is the cross-sectional area of the tube and R is the resistance.

***2. The radiation impedance* acts** whenever sound radiates from some element and is made up of an acoustic mass associated with accelerating the air particles near the surface of the element and a resistance associated with the transmission of sound energy into the far field. Since the volume velocities associated with these two processes add (some fraction of U goes into accelerating the mass layer, while the rest radiates away from the element), we can think of these as two parallel elements.

Radiation from the end of an organ pipe of radius a can be modeled by the following:



where:

$$\frac{\underline{U}}{\underline{P}} = \underline{Y}_{Rad} = \frac{1}{\underline{Z}_{Rad}} = \frac{1}{j\omega L_R} + \frac{1}{R_R}$$

$$= \frac{1}{\frac{j\omega 0.8a}{2\pi a^2}} + \frac{1}{\frac{\rho_0 c}{2\pi a^2}}$$

Note that the radiation mass is equivalent to the addition of a tube of radius a and length 0.8a to the end of the pipe. *This is the end correction!!*

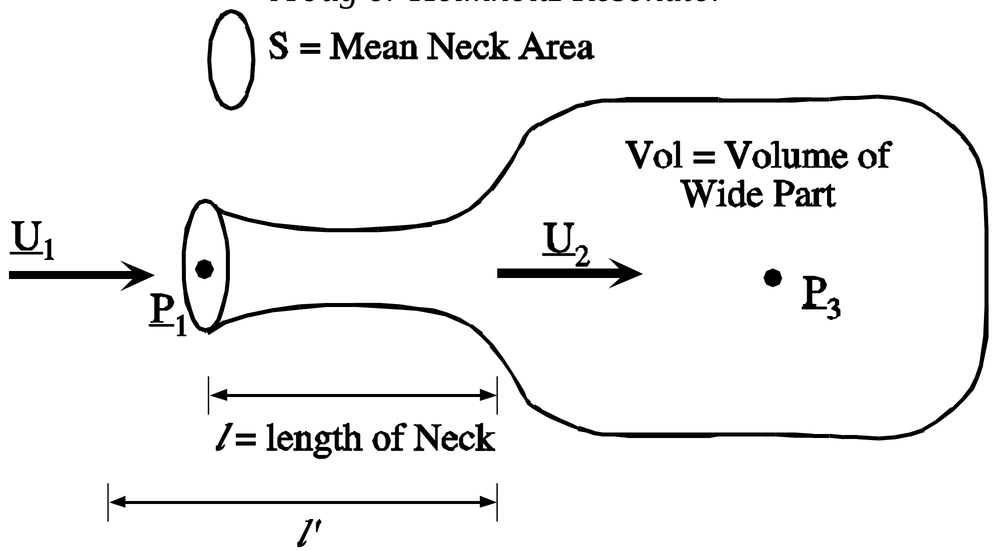
E. Range of applicability of acoustic circuit theory.

1. Pressure and volume-velocity ranges consistent with “linear acoustics”.

2. Frequency range limited by the assumption of “lumped” elements, i.e. the dimensions of the structures need to be small compared to a wavelength: a and $l < 0.1\ \lambda$.

VI. Circuit Descriptions of a Real Acoustic System

A Jug or Helmholtz Resonator

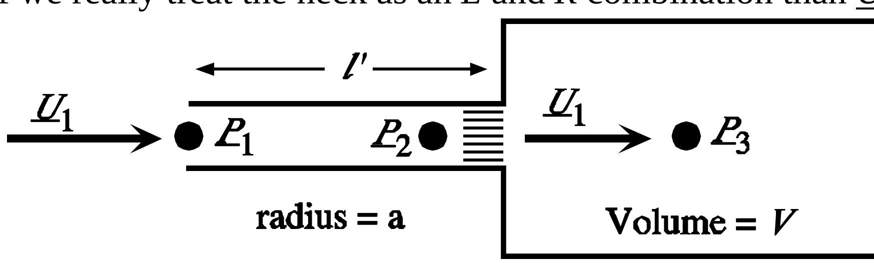


A. An Acoustic Circuit Description

If we are using acoustic volume velocity as a through variable; the flow of volume velocity through the neck suggests a series combination of Acoustic Elements. The volume velocity first flows through an a series combination of an acoustic inertance L_A, and an acoustic resistor R_A, and then into the acoustic compliance C_A of the closed cavity, where:

$$L_A = \frac{\rho_0 l'}{\pi a^2};\ R_A = g(l, a, frequency);\ C_A = \frac{Volume}{\gamma P_0}.$$

Furthermore if we really treat the neck as an L and R combination than $\underline{U}_2 = \underline{U}_1$.



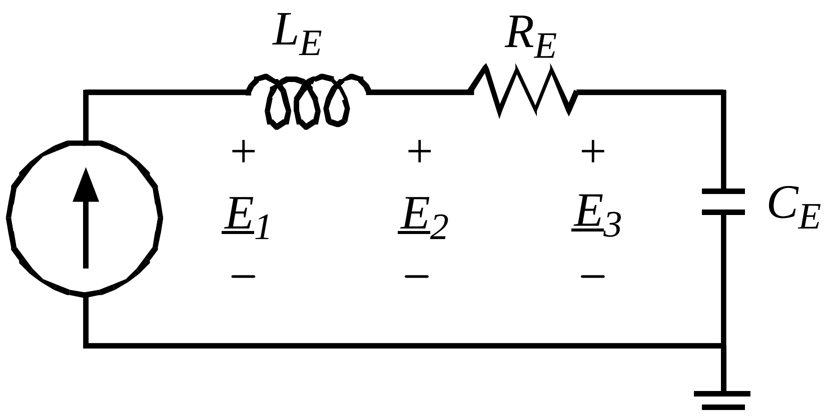
In the sinusoidal steady state: $\underline{P}_1(\omega) = \underline{U}_1(\omega)\left(j\omega L_A + R_A + \frac{1}{j\omega C_A}\right)$

The ratio of P_1/U_1 defines the acoustic input impedance of the bottle and in this case it is equal to the series sum of the impedance of the three series elements.

$$\underline{Z}_{IN}(\omega) = j\omega L_A + R_A + \frac{1}{j\omega C_A}$$

B. An Electrical Analog of the Acoustic Circuit Description

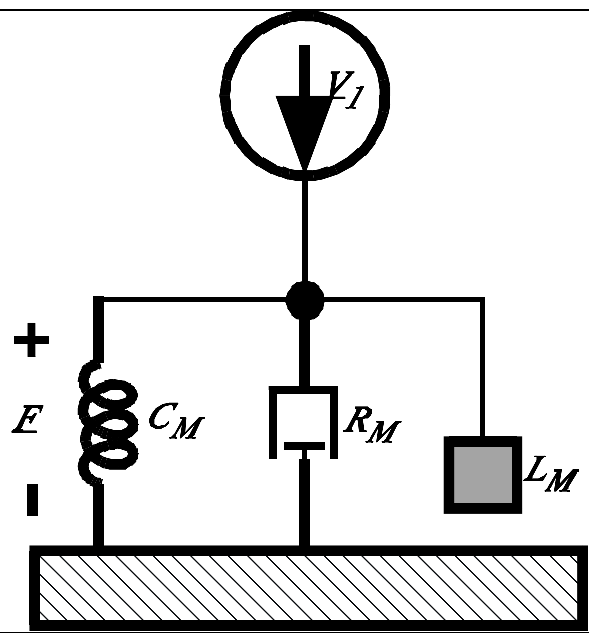
In Electrical circuits the wires that connect the ideal elements are perfect conductors.



If the numerical values of $L_E = L_A$, $R_E=R_A$, and $C_E=C_A$, *then* $I_1=U_1$, $E_1=P_1$ and $E_2=P_2$.

C. A Mechanical Analog of the Acoustic Circuit Description

In Mechanical circuits the rods that attach ideal mechanical elements are rigid and massless.



If the numerical values of $L_M = L_A$, $R_M=R_A$, $C_M=C_A$, then $\underline{V}_1=\underline{U}_1=\underline{U}_2$, and $\underline{F}=\underline{P}_1$, then

$$\frac{\underline{F}}{\underline{V}_1} = j\omega L_M + R_M + \frac{1}{j\omega C_M}$$

Where the total force acting on the elements equals the sum of the forces acting on each.

$$\underline{F} = \underline{V}_1\left(j\omega L_M + R_M + \frac{1}{j\omega C_M}\right)$$