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Notes on Acoustic Circuits
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LECTURE 9

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6 Equivalent Circuits

Rather than present methods for solving the equations that govern arbitrary acoustic networks, the sections below develop general techniques for analyzing an important subclass: networks made up of sources and *linear elements*.

6.1 Superposition - the Effects of Multiple Sources.

The acoustic circuit elements introduced so far are of two types: sources, which specify either the element pressure or volume velocity in terms of an intrinsic property of the source, and (in the case of pressures and volume velocities that have an e^{st} time dependence) impedances for which the amplitude of the pressure and volume velocity are proportional. The equations that govern the element pressures and volume velocities can be written so that the unknown element pressures and volume velocities occur on the left side, with constant (integers or impedances) factors and the intrinsic pressures and volume velocities of the sources appear on the right. These equations are *linear* in terms of the unknown pressures and volume velocities. According to the theory of linear equations, each of the unknown element pressures

[2]The consequences of equality will be discussed later.

and volume velocities (e.g., b_i) can be expressed as a linear weighted sum of the intrinsic pressures and volume velocities of the sources:

$$b_i = \sum_{j=1}^{M} \alpha_j P_{S_j} + \sum_{k=1}^{N} \beta_k U_{S_k}$$

In circuit theory, this result is known as the Superposition Principle. This Principle allows circuits with multiple sources to be analyzed simply by considering the effects of each source separately. In combination, the contributions of the sources simply add. Importantly, when each source is considered separately, impedances are often found to be connected in series or in parallel.

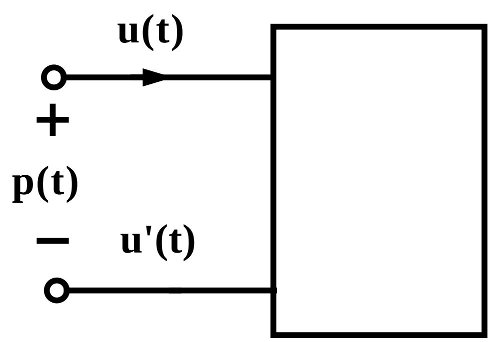

Figure 10: Example of an acoustic circuit containing a port. It can be shown that Kirchoff's volume velocity law requires that $u(t) = u'(t)$.

6.2 Equivalent Acoustic Circuits for One-Ports.

A *port* (Fig. 10) is a pair of terminals (or nodes) in an acoustic circuit that can be connected to another circuit.

6.2.1 Thevénin Equivalent Circuits

If the "other" circuit is a volume velocity source of arbitrary intrinsic velocity U_0, then it can be shown that the pressure P across the terminals of the port must satisfy

$$P = P_{Th} + Z_{Th} U_0 \tag{9}$$

This result was first obtained by the French telegrapher Thevénin and is usually referred to as *Thevénin's Theorem.*

In Eq. 9, P_{Th} has a non-zero value only when the sources in the acoustic circuit (other than U_0) have non-zero intrinsic values, and is said to be the "open-circuit" (i.e., when $U_0 = 0$) pressure of the circuit to the right of the terminals. If all sources in the circuit to the right of the terminals have zero intrinsic value, $P = Z_{Th} U_0$. For this reason Z_{Th} is

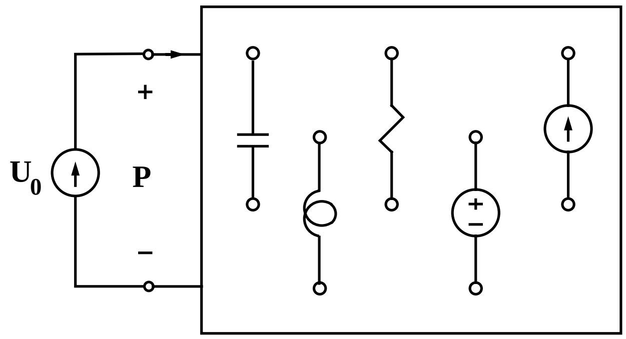

Figure 11: An acoustic circuit for which it is possible to apply Thevénin's Theorem.

said to be the equivalent impedance (or internal impedance) of the circuit to the right of the terminals. This generalizes the notion of impedances equivalent to the connection of two impedances in series or in parallel.

It can be shown that if Thevénin's Theorem applies at a pair of terminals, the dependence of P on U_0 in the acoustic circuit of Fig. 11 is the same as for the acoustic circuit of Fig. 12. The series connection of Z_{Th} and P_{Th} is said to be the "Thevénin Equivalent" of the acoustic circuit to the right of the terminals in Fig. 11.

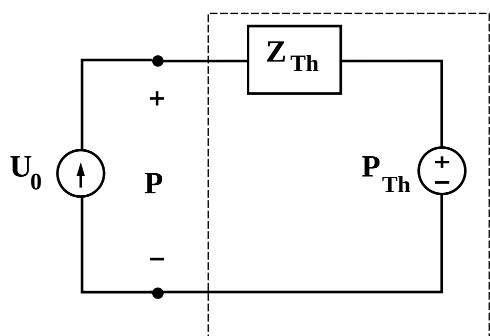

Figure 12: The "Thevénin Equivalent" acoustic circuit at a pair of terminals.

6.2.2 Norton Equivalent Circuits

Alternatively, if the "other" circuit is a pressure source of arbitrary intrinsic pressure P_0, then it can be shown that the volume velocity entering the terminal of the port must satisfy

$$U = -U_N + \frac{P_0}{Z_N} \tag{10}$$

This result was first obtained by an engineer (and M. I. T. graduate) at the Bell Telephone Laboratories and is usually referred to as *Norton's Theorem.*

It can be shown that if Norton's Theorem applies at a pair of terminals, the dependence of U on P_0 in the acoustic circuit of Fig. 13 is the same as for the acoustic circuit of Fig. 14.

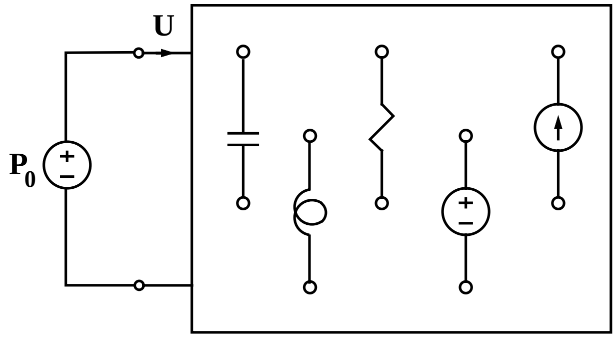

Figure 13: An acoustic circuit for which it is possible to apply Norton's Theorem.

The parallel connection of Z_N and U_N is said to be the "Norton Equivalent" of the acoustic circuit to the right of the terminals in Fig. 13.

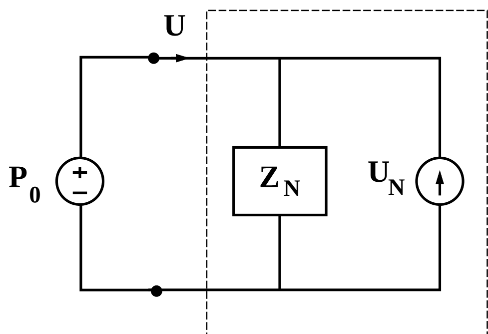

Figure 14: The Norton Equivalent acoustic circuit at a pair of terminals.

In addition to simplifying the analysis of acoustic circuits, Thev́enin's Theorem can be used to develop circuit models for imperfect sound sources (as opposed to the "ideal" volume velocity and pressure sources introduced in Secs. 3.1 and 3.2). An imperfect (real world) volume velocity source can be represented as an ideal volume velocity source in parallel with an impedance. An imperfect pressure source can be represented as an ideal pressure source in series with an impedance.

6.2.3 Thev́enin - Norton Relations

In many acoustic circuits, it is possible to connect either a pressure source of arbitrary intrinsic pressure or a volume velocity source of arbitrary intrinsic velocity at the same pair of terminals. Such circuits satisfy both Thev́enin's Theorem and Norton's Theorem at those terminals. For such circuits, the Theévenin and Norton Equivalents are related:

$$P_{Th} = U_N Z_N \quad (11)$$

$$Z_{Th} = Z_N \quad (12)$$

6.3 Equivalent Acoustic Circuits for Two-Ports.

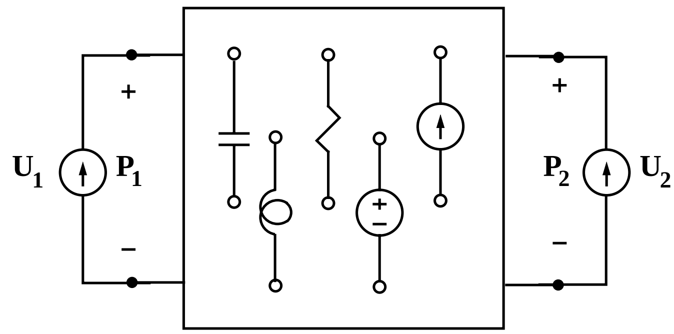

Figure 15: An acoustic circuit to which it is possible to connect two one-port circuits. In this case the one-port circuits are volume velocity sources with arbitrary intrinsic velocities U_1 and U_2.

The concept of equivalent circuits can be generalized to acoustic circuits that contain more than one port. When there are two ports in an acoustic circuit, it is necessary to distinguish between cases in which it is possible to connect two arbitrary volume velocity sources (e.g., Fig. 15), two arbitrary pressure sources, and one arbitrary volume velocity source and one arbitrary pressure source. The concept of two-ports generalizes readily to N-ports, where $N \geq 2$. Such circuits are used

- To develop simpler or abstract representations of portions of circuits.
- To define abstract elements, e.g. acoustic-mechanical transformers.

In the first case, e.g. Fig. 15, a generalization of Thevénin's Theorem is applicable:

$$P_1 = Z_{11}U_1 + Z_{12}U_2 + P_{OC_1} \tag{13}$$
$$P_2 = Z_{21}U_1 + Z_{22}U_2 + P_{OC_2} \tag{14}$$

In the second case, a generalization of Norton's Theorem is applicable:

$$U_1 = Y_{11}P_1 + Y_{12}P_2 - U_{SC_1} \tag{15}$$
$$U_2 = Y_{21}P_1 + Y_{22}P_2 - U_{SC_2} \tag{16}$$

When it is possible to connect an arbitrary volume velocity source at the first port and an arbitrary pressure source at the second:

$$P_1 = S_{11}U_1 + S_{12}P_2 + P_{OS_1} \tag{17}$$
$$U_2 = S_{21}U_1 + S_{22}P_2 - U_{OS_2} \tag{18}$$

As in the case of one-ports, the terms P_{OC_1}, P_{OC_2}, U_{SC_1}, U_{SC_2}, P_{OS_1}, and U_{OS_2} represent the effects of sources within the two-port. If there are no sources in the two-port, or if the intrinsic values of the sources in the two-port are all zero, these terms are necessarily zero.

In the absence of sources in the two-ports, the above relations become

$$P_1 = Z_{11}U_1 + Z_{12}U_2 \tag{19}$$
$$P_2 = Z_{21}U_1 + Z_{22}U_2 \tag{20}$$

In the second case, a generalization of Norton's Theorem is applicable:

$$U_1 = Y_{11}P_1 + Y_{12}P_2 \tag{21}$$
$$U_2 = Y_{21}P_1 + Y_{22}P_2 \tag{22}$$

When it is possible to connect an arbitrary volume velocity source at the first port and an arbitrary pressure source at the second:

$$P_1 = S_{11}U_1 + S_{12}P_2 \tag{23}$$
$$U_2 = S_{21}U_1 + S_{22}P_2 \tag{24}$$

6.3.1 Reciprocity

Acoustic circuits composed of volume velocity sources, pressure sources, and impedances satisfy reciprocity relations that are easily expressed in terms of the two-port descriptions. For example, in Eq. 13 and 14, $Z_{12} = Z_{21}$, in Eq. 15 and 16, $Y_{12} = Y_{21}$, and in Eq. 17 and 18, $S_{12} = -S_{21}$.

In the absence of sources reciprocity ensures that, corresponding to Eq. 19 and 20,

$$\left.\frac{P_1}{U_2}\right|_{U_1=0} = \left.\frac{P_2}{U_1}\right|_{U_2=0} \tag{25}$$

corresponding to Eq. 21 and 22,

$$\left.\frac{U_1}{P_2}\right|_{P_1=0} = \left.\frac{U_2}{P_1}\right|_{P_2=0} \tag{26}$$

and corresponding to Eq. 23 and 24,

$$\left.\frac{P_2}{P_1}\right|_{U_2=0} = -\left.\frac{U_1}{U_2}\right|_{P_1=0}. \tag{27}$$

These consequences of reciprocity can be observed in sound fields as well as in acoustic circuits.

7 System Functions

When the pressures and volume velocities in an acoustic circuit all have the same exponential time behavior, ratios of the amplitudes of pressures across node pairs, or the amplitudes of volume velocities through elements, to the the amplitudes of the intrinsic pressures of pressure sources or of the intrinsic velocities of volume velocity sources, are commonly called *system functions*.

For example, in a one-port that contains no sources, (e.g., Fig. 11), the ratio

$$\frac{P}{U_0} = Z_{Th}(s).$$

is said to be the *driving point impedance* system function. Similarly, in the two port of Fig. 15, the ratio

$$\left.\frac{P_2}{U_1}\right|_{U_2=0} = Z_{21}(s).$$

is said to be the transfer impedance system function for the two ports. In each case, the notation emphasizes the dependence of the ratio on the frequency s of the source.

7.1 Acoustic Circuits

For acoustic circuits consisting of a finite number of elements, system functions take the form of a ratio of polynomials in s, e.g.

$$H(s) = \frac{n(s)}{d(s)} = K\frac{s^M + b_1 s^{M-1} + \cdots + b_M}{s^N + a_1 s^{N-1} + \cdots + a_N},$$

where K, the a_i, and the b_j are real numbers.

The fundamental theorem of algebra states that each of the polynomials $n(s)$ and $d(s)$ may be represented as a product of factors

$$H(s) = K\frac{(s - z_1)(s - z_2)\cdots(s - z_M)}{(s - p_1)(s - p_2)\cdots(s - p_N)}.$$

The z_i are said to be the *zeroes* of the system function $H(s)$ and the p_j are said to be the *poles* of $H(s)$. In general, these zeroes and poles are *complex* numbers.

The interpretation of the z_i is straightforward. As $s \to z_i$, $H(s) \to 0$. For the poles, on the other hand, as $s \to z_j$, $H(s) \to \infty$. For these values of s it is possible to have non-zero pressures and volume velocities in an acoustic circuit when all sources have zero value. The poles are often called the *natural frequencies* of the circuit (see Sec. 8.3).

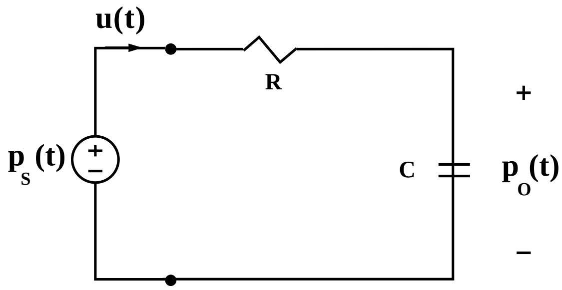

Figure 16: Circuit for Example 1. Assume that all pressures and volume velocities have an exponential (e^{st}) time dependence.

7.2 Example 1 – Resistance-Compliance Circuit

a) Determine the input admittance system function ($Y(s) = U/P_S$) and the pressure transfer ratio system function ($H(s) = P_O/P_S$) for the circuit of Fig. 16.

b) Identify the values of the poles and zeroes of $Y(s)$ and $H(s)$

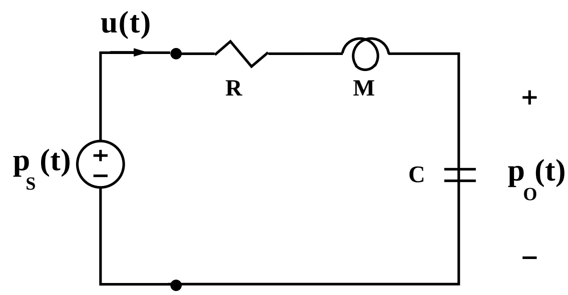

Figure 17: Circuit for Example 2. Assume that all pressures and volume velocities have an exponential (e^{st}) time dependence.

7.3 Example 2 – Resistance-Mass-Compliance Circuit

a) Determine the input admittance system function ($Y(s) = U/P_S$) and the pressure transfer ratio system function ($H(s) = P_O/P_S$) for the circuit of Fig. 17.

b) Identify the values of the poles and zeroes of $Y(s)$ and $H(s)$

8 The Sinusoidal Steady State

Understanding the behavior of acoustic circuits when the intrinsic pressures and volume velocities of sources vary sinusoidally in time is of considerable theoretical and practical importance. If the circuits contain finite positive acoustic resistances, the transient pressures and volume velocities that may be excited when the sources are first turned on eventually die away while the sinusoidal pressures and volume velocities corresponding to the sources continue and eventually dominate the response to the sources. The importance of understanding the response to sinusoidal sources reflects the following:

- Arbitrarily complex periodic time waveforms can be represented as an infinite "Fourier Series", a sum of terms each of which has a sinusiodal time dependence.[3] In a circuit composed of linear elements, the response to a sum of sinusoidal waveforms is the sum of the responses to each component separately.

- In the sinusoidal steady state, the pressures and volume velocities elicited in a circuit by sources having sinusoidal waveforms have sinusiodal waveforms with the same frequency as the sources. These can be characterized simply in terms of their relative amplitudes and phase angles. The dependence of the relative amplitudes and phase angles on frequency provides a complete description of the relation between the sources and the pressures and volume velocities.

- The use of sinusoidal sources greatly simplifies the task of making measurements to characterize the behavior of real acoustic systems. In addition to the properties mentioned above, use of sinusoids permits measurements to be made at any time after the transient components of the responses have died away.

8.1 Source - Response Relations

Assume that all pressures and volume velocities in a circuit have an e^{st} time dependence, and that when the source waveform is $x(t) = Xe^{st}$, the pressure across a pair of nodes or volume velocity through an element is $y(t) = Ye^{st}$, where the system function relating Y to X is

$$H(s) = \frac{Y}{X}$$

When $s = j\omega$, H is typically a complex quantity. In general, X and Y may be complex as well. To emphasize this, bold symbols are used to denote complex quantities, thus

$$\mathbf{H}(j\omega) = \frac{\mathbf{Y}}{\mathbf{X}}$$

[3]This result can be generalized to non-periodic waveforms, in which case the infinite series becomes an integral.

In the sinusoidal steady state

$$\begin{aligned}
x(t) &= X\cos(\omega t) = \mathrm{Re}\left[Xe^{j\omega t}\right] = \mathrm{Re}\left[\mathbf{X}e^{j\omega t}\right] \\
y(t) &= Y\cos(\omega t+\theta) = \mathrm{Re}\left[Ye^{j(\omega t+\theta)}\right] = \mathrm{Re}\left[\mathbf{Y}e^{j\omega t}\right].
\end{aligned}$$

Where $\mathbf{X} = X$ and $\mathbf{Y} = Ye^{j\theta} = \mathbf{H}(j\omega)\mathbf{X}$. For each value of ω, $\mathbf{H}$ is a complex number

$$\mathbf{H}(j\omega) = |\mathbf{H}|\, e^{j\theta_H}$$

where

$$\begin{aligned}
|\mathbf{H}| &= \sqrt{(\mathrm{Re}\,[\mathbf{H}])^2 + (\mathrm{Im}\,[\mathbf{H}])^2} \\
\theta_H &= \arctan\frac{\mathrm{Im}\,[\mathbf{H}]}{\mathrm{Re}\,[\mathbf{H}]}.
\end{aligned}$$

It follows from Eq. 57 that

$$|\mathbf{Y}| = |\mathbf{H}|\,|\mathbf{X}| = |\mathbf{H}|\,X \quad (28)$$

$$\theta = \theta_H, \quad (29)$$

so that

$$\begin{aligned}
y(t) &= \mathrm{Re}\left[\mathbf{X}\,|\mathbf{H}|\,e^{j\theta_H}e^{j\omega t}\right] \\
&= X\,|\mathbf{H}|\cos(\omega t+\theta_H). \quad (30)
\end{aligned}$$

This illustrates the central role played by the magnitude and angle of system functions when $s = j\omega$.

8.1.1 Example 1 – Resistance-Compliance Circuit

The system functions for the circuit of Fig. 16 can be shown to be of the form:

$$\begin{aligned}
H(s) = \tfrac{P_O}{P_S} &= \frac{\omega_0}{\omega_0 + s} \\
Y(s) = \tfrac{U}{P_S} &= \frac{1}{R}\frac{s}{\omega_0 + s},
\end{aligned}$$

where $\omega_0 = 1/RC$. It is easy to see that as $s \to 0$, $H(s) \to 1$ and $Y(s) \approx s/\omega_0 \to 0$. Similarly, as $s \to \infty$, $H(s) \approx \omega_0/s \to 0$ and $Y(s) \to 1/R$. These observations are consistent with a physical understanding of the circuit of Fig. 16. As $s \to 0$, the impedance of the compliance becomes arbitrarily large, the impedance seen across the terminals becomes arbitrarily large, and the pressure division ratio approaches unity. As $s \to \infty$, the impedance of the compliance approaches 0, the impedance seen across the terminals approaches R, and and the pressure division ratio approaches 0. When $s = j\omega$,

$$\mathbf{H}(j\omega) = \frac{\omega_0}{\omega_0 + j\omega} = \frac{1}{1 + j\Omega}, \quad (31)$$

where $\Omega = \omega/\omega_0$ is the *normalized frequency.* Making use of Eq. 49, 50, and 58, one has

$$|\mathbf{H}(j\omega)| = \frac{1}{\sqrt{1+\Omega^2}} \quad (32)$$

$$\theta_H = -\arctan\Omega \quad (33)$$

Because $s = 0$ only when $\omega = 0$ and $\omega \to \infty$ implies $s \to \infty$, these expressions confirm the limiting behavior of H as $\omega \to 0$ and as $\omega \to \infty$. But they also show that both $|\mathbf{H}|$ and θ_H decrease monotonically as ω increases. In the special case $\omega = \omega_0$, $\Omega = 1$, $|\mathbf{H}| = 1/\sqrt{2} = 0.707\ldots$ and $\theta_H = -45^o$. Also,

$$\mathbf{Y}(j\omega) = \frac{1}{R}\frac{j\omega}{\omega_0 + j\omega} = \frac{1}{R}\frac{j\Omega}{1+j\Omega}. \quad (34)$$

Note that $\mathbf{Y}(j\omega)$ is just $\mathbf{H}(j\omega)$ multiplied by $j\Omega/R$. Since

$$\frac{j\Omega}{R} = \frac{\Omega}{R}e^{j\frac{\pi}{2}},$$

$$\mathbf{Y}(j\omega) = \frac{\Omega}{R}\mathbf{H}(j\omega)e^{j\frac{\pi}{2}}$$

making use of Eq. 49, 50, and 57, one has

$$|\mathbf{Y}(j\omega)| = \frac{1}{R}\frac{\Omega}{\sqrt{1+\Omega^2}} \quad (35)$$

$$\theta_Y = \frac{\pi}{2} - \arctan\Omega \quad (36)$$

These expressions show that while θ_Y decreases monotonically as ω increases, $|\mathbf{Y}|$ increases monotonically as ω increases. In the special case $\omega = \omega_0$, $\Omega = 1$, $|\mathbf{Y}| = 1/(\sqrt{2}R) \approx 0.707/R$ and $\theta_H = +45^o$. The dependence of the magnitude and angle of H and Y on Ω are illustrated in Fig. 18. The system function H is said to have a *lowpass* characteristic because $|\mathbf{H}| \to 1$ as $\omega \to 0$ and $|\mathbf{H}| \to 0$ as $\omega \to \infty$. Similarly Y is said to have a *highpass* characteristic because $|\mathbf{Y}| \to 0$ as $\omega \to 0$ and $|\mathbf{Y}| \to 1$ as $\omega \to \infty$.

8.1.2 Example 2 – Resistance-Mass-Compliance Circuit

The system functions for the circuit of Fig. 17 are:

$$H(s) = \tfrac{P_O}{P_S} = \frac{\omega_0^2}{s^2 + \alpha\omega_0 s + \omega_0^2}$$

$$Y(s) = \tfrac{U}{P_S} = \frac{1}{M}\frac{s}{s^2 + \alpha\omega_0 s + \omega_0^2},$$

where $\omega_0 = 1/\sqrt{MC}$, $\Omega = \omega/\omega_0$ is the *normalized frequency,* and $\alpha = R/\sqrt{M/C}$ is the *normalized resistance.*[4]

[4]The quantity $Z_C = \sqrt{M/C}$ is the characteristic impedance of the circuit.

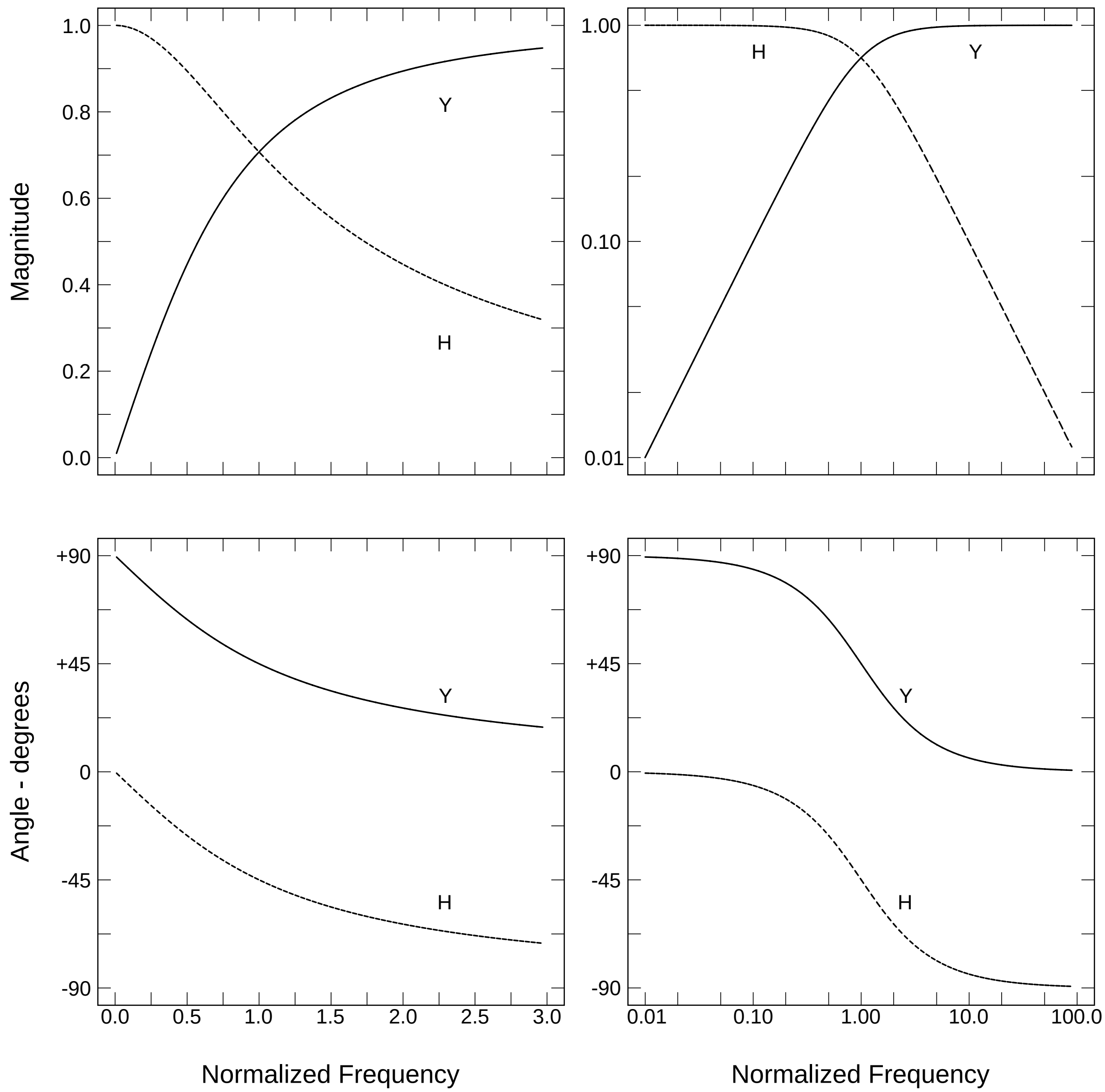

Figure 18: Dependence of the magnitude and phase of the input admittance system function Y (solid curves) and pressure transfer ratio system function H (dotted curves) on normalized frequency $\Omega = \omega/\omega_0$ for the acoustic circuit of Fig. 16. The magnitude of the admittance system function Y has been multiplied by R.

It is easy to see that as $s \to 0$, $H(s) \to 1$ and $Y(s) \approx s/M \to 0$. Similarly, as $s \to \infty$, $H(s) \approx (\omega_0/s)^2 \to 0$ and $Y(s) \approx 1/(Ms) \to 0$. When $s = j\omega$,

$$\mathbf{H}(j\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j\alpha\omega_0\omega} = \frac{1}{(1-\Omega^2) + j\alpha\Omega}, \tag{37}$$

Note that although Eq. 37 bears a certain resemblance to Eq. 31, the real part of the denominator is not constant. In particular it vanishes when $\Omega = 1$, i.e., when $\omega = \omega_0$. Making use of Eq. 49, 50, and 58, one has

$$|\mathbf{H}(j\omega)| = \frac{1}{\sqrt{(1-\Omega^2)^2 + (\alpha\Omega)^2}} \tag{38}$$

$$\theta_H = -\arctan\frac{\alpha\Omega}{1-\Omega^2} \tag{39}$$

Note that when $\omega = \omega_0$, $\Omega = 1$, $|\mathbf{H}| = 1/\alpha$ and $\theta_H = -90^o$. Also,

$$\mathbf{Y}(j\omega) = \frac{1}{M}\frac{j\omega}{(\omega_0^2 - \omega^2) + j\alpha\omega_0\omega} = \frac{1}{Z_C}\frac{j\Omega}{(1-\Omega^2) + j\alpha\Omega}. \tag{40}$$

Similar to the RC circuit, $\mathbf{Y}$ is just $\mathbf{H}$ multiplied by $j\Omega/Z_C$. Making use of Eq. 49, 50, and 57, one has

$$|\mathbf{Y}(j\omega)| = \frac{1}{Z_C}\frac{\Omega}{\sqrt{(1-\Omega^2)^2 + (\alpha\Omega)^2}} \tag{41}$$

$$\theta_Y = \frac{\pi}{2} - \arctan\frac{\alpha\Omega}{1-\Omega^2} \tag{42}$$

Note that, when $\omega = \omega_0$, $\Omega = 1$, $|\mathbf{Y}| = 1/(\alpha Z_C) = 1/R$ and $\theta_H = 0^o$, so that from the point of view of the terminals, the RMC circuit is indistinguishable from a resistance of value R.

The dependence of the magnitude and angle of H and Y on Ω are illustrated in Fig. 19. The system function H is said to have a *lowpass* characteristic because $|\mathbf{H}| \to 1$ as $\omega \to 0$ and $|\mathbf{H}| \to 0$ as $\omega \to \infty$. On the other hand, Y is said to have a *bandpass* characteristic because $|Y| \to 0$ as $\omega \to 0$ and also as $\omega \to \infty$, but $|\mathbf{Y}| = 1/R > 0$ when $\omega = \omega_0$. Note that, as ω increases, both θ_H and θ_Y decrease over a range of 180^o, twice the range for the RC circuit of Fig. 18. Also the entire decrease of these angles is largely confined to a small range of frequencies near $\omega = \omega_0$ where $|\mathbf{H}|$ and $|\mathbf{Y}|$ exhibit sharp peaks. This pairing of highly peaked magnitude functions and phase angles that change rapidly over nearly 180^o is characteristic of a highly-tuned *resonant* circuit.

8.2 Generalization

The system function notion can be applied to a wider class of systems than circuits with a finite number of elements. These systems are generally called *linear, time independent*

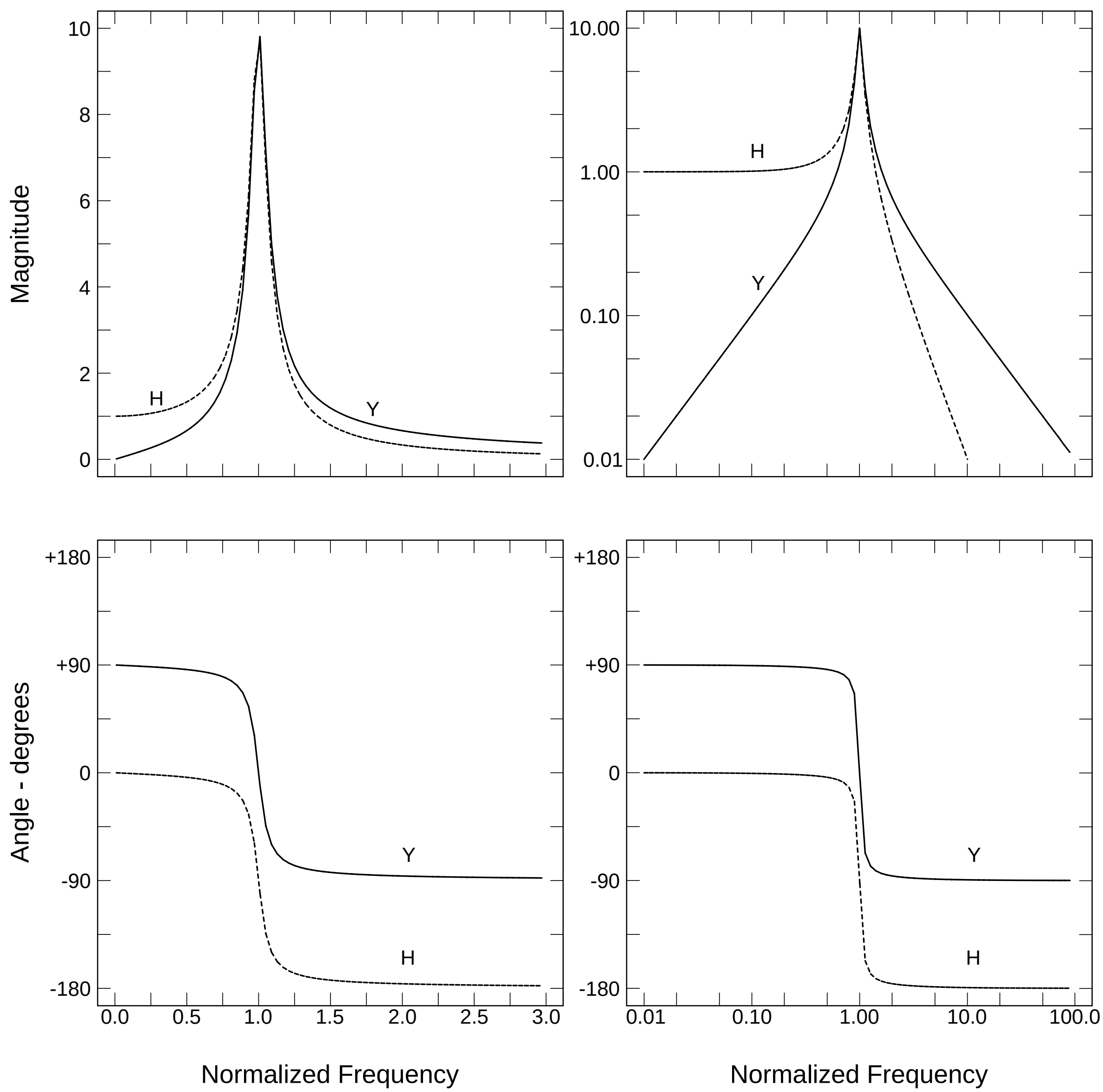

Figure 19: Dependence of the magnitude and phase of the input admittance system function Y (solid curves) and pressure transfer ratio system function H (dotted curves) on normalized frequency $\Omega = \omega/\omega_0$ for the acoustic circuit of Fig. 17. The magnitude of the admittance system function Y has been multiplied by Z_C. The curves have been drawn for the case $\alpha = 0.1$.

systems, and include, for example, acoustic tubes. The volume velocity transfer function for an acoustic tube of length L that is closed at one end and open at the other is

$$\mathbf{H}(j\omega) = \frac{U}{U_S} = \frac{1}{\cos\omega\frac{L}{c}}$$

Clearly $H(s)$ cannot be written as a ratio of polynomials in the variable s, because $\cos x$ has an infinite number of zeroes. However

$$\mathbf{H}(j\omega) = \prod_{k=1}^{\infty} \frac{\omega_k^2}{\omega_k^2 - \omega^2}$$

where

$$\omega_k = 2\pi k \frac{c}{4L}$$

and $k = 1, 3, \ldots$.

8.3 Example - Natural Frequencies

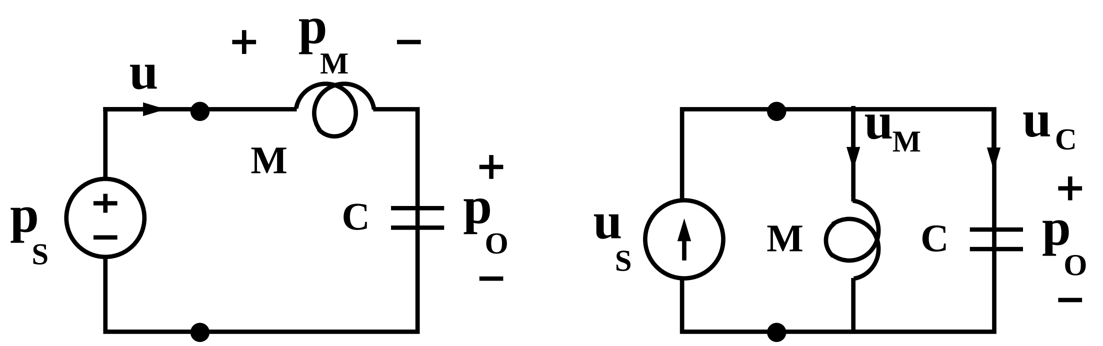

Figure 20: Circuits that have a single acoustic mass and a single acoustic compliance. In the circuit on the left, all elements are connected in series; in that on the right all elements are connected in parallel.

Consider the circuits of Fig. 20. When all pressures and volume velocities have an e^{st} time dependence, the system functions for the series circuit can be shown to be

$$\begin{aligned} H &= \frac{P_O}{P_S} = \frac{\omega_0^2}{s^2 + \omega_0^2} \\ Y &= \frac{U}{P_S} = \frac{1}{M}\frac{s}{s^2 + \omega_0^2}, \end{aligned}$$

where $\omega_0 = 1/\sqrt{MC}$. Similarly for the parallel circuit

$$Z = \frac{P_O}{U_S} = \frac{1}{C}\frac{s}{s^2 + \omega_0^2}.$$

Note that the poles of all three system function are the same:

$$p_i = \pm j\omega_0$$

To understand why these poles are the natural frequencies of these circuits, consider the acoustic circuit of Fig. 21, which is identical to the circuits of Fig. 20 when the sources have zero intrinsic value.

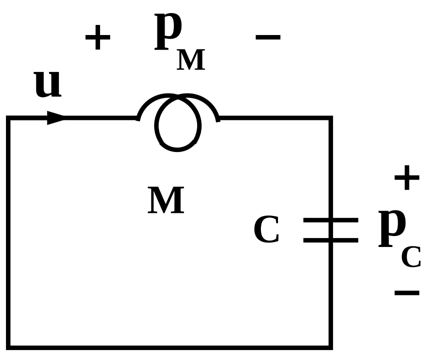

Figure 21: The circuits of Fig. 20 reduce to this two element circuit when the sources have zero intrinsic value.

For this circuit, the elements are connected both in series and in parallel. Kirchoff's Volume Velocity Law requires that both elements have the same volume velocity, $u(t)$. Kirchoff's Pressure Law requires that $p_M = -p_C$ or

$$M\frac{du(t)}{dt} = -p_C(t),$$

while the defining property of a compliance requires that

$$u(t) = C\frac{dp_C(t)}{dt}.$$

Differentiating the latter equation with respect to time yields an expression for the derivative of volume velocity that can be substituted in the former equation:

$$MC\frac{d^2p_C(t)}{dt^2} + p_C(t) = 0,$$

or

$$\frac{d^2p_C(t)}{dt^2} + \omega_0^2 p_C(t) = 0.$$

Second order linear differential equations of this type have been encountered when solving the one-dimensional acoustic wave equation in the case of sinusoidal time dependence. The solution is readily seen to be of the form

$$p_C(t) = P_C \cos(\omega t + \theta)$$

because the time derivative of this expression is

$$\frac{dp_C}{dt} = -\omega_0 \sin(\omega t + \theta)$$

and the second derivative is

$$\frac{d^2 p_C(t)}{dt^2} = -\omega_0^2 \cos(\omega t + \theta) = -\omega_0^2 p_C(t)$$

This demonstrates that it is possible to have pressures and volume velocities in an acoustic circuit with no sources provided their frequencies are the values of the poles of the system function. In the case considered, the poles (and natural frequencies) are purely imaginary (simplifying the algebra) so the pressures and volume velocities have a sinusoidal time dependence. In the more general case, the poles would be complex, with negative real parts, and the time dependence associated with the complex natural frequencies would be exponentially decaying sinusoids.

9 Power and Energy

This section derives three results relevant to power and energy in acoustic circuits. Although the results are obtained for the series RMC circuit of Fig. 22, they apply to arbitrary connections resistances, masses, and compliances

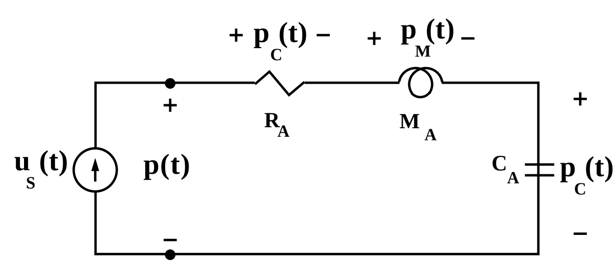

Figure 22: Example used to analyze power and energy in acoustic circuits.

In the acoustic circuit of Fig. 22 all elements have the same volume velocity $u_S(t)$ and Kirchoff's Pressure Law requires

$$p(t) = p_R(t) + p_M(t) + p_C(t)$$

The power that the volume velocity source supplies to the circuit to the right of the terminals is thus

$$w(t) = p(t)u_S(t) = p_R(t)u_S(t) + p_M(t)u_S(t) + p_C(t)u_S(t)$$

but since

$$\begin{aligned} p_R(t) &= R_A u_S(t) \\ p_M(t) &= M_A \frac{du_S(t)}{dt} \\ u_S(t) &= C_A \frac{dp_C(t)}{dt} \end{aligned}$$

$$\begin{aligned} w(t) &= R_A u_S^2(t) + M_A u_S(t) \frac{du_S(t)}{dt} + +C_A p_C(t) \frac{dp_C(t)}{dt} \\ w(t) &= R_A u_S^2(t) + \frac{dE_M(t)}{dt} + \frac{dE_C(t)}{dt} \end{aligned}$$

Since $R_A u_S^2(t)$ is the power absorbed by the acoustic resistance $w_d(t)$, E_M is the energy stored in the acoustic mass, and E_C is the energy stored in the acoustic compliance, one has

$$w(t) = w_d(t) + \frac{d\left(E_M(t) + E_C(t)\right)}{dt}, \tag{43}$$

that is, the power flowing into the terminals from the source is equal to the power dissipated in the acoustic resistance plus the rate of increase of energy stored in the mass and compliance elements.

9.1 Sinusoidal Steady State

Assume that in the sinusoidal steady state

$$\begin{aligned} u_S(t) &= U_S \cos(\omega t + \phi) &= \text{Re}\left[\mathbf{U_S} e^{j\omega t}\right] \\ p(t) &= P \cos(\omega t + \theta) &= \text{Re}\left[\mathbf{P} e^{j\omega t}\right]. \end{aligned}$$

Then

$$w(t) = U_S P \cos(\omega t + \phi) \cos(\omega t + \theta).$$

Making use of the trigonometric identity

$$\cos x \cos y = \frac{1}{2}\cos(x - y) + \frac{1}{2}\cos(x + y)$$

$$w(t) = \frac{1}{2} U_S P \cos(\theta - \phi) + \frac{1}{2} U_S P \cos(2\omega t + \theta + \phi)$$

This can be readily shown to equal

$$w(t) = \frac{1}{2}\text{Re}\left[\mathbf{P}\mathbf{U_S^*}\right] + \frac{1}{2}|\mathbf{P}|\,|\mathbf{U_S}| \cos(2\omega t + \theta + \phi)$$

The average value of the second term over a period ($T = 2\pi/\omega$) is zero. The average value of the power supplied to the circuit to the right of the terminals in Fig. 22 is

$$W_{\text{av}} = \frac{1}{2}\text{Re}\left[\mathbf{P}\mathbf{U}^*\right] \tag{44}$$

Note that since $\mathbf{P}\mathbf{U}^* = \mathbf{P}^*\mathbf{U}$ the choice for defining W_{av} in Eq. 44 is somewhat arbitrary. The reason for this choice is explained below.

9.2 Vector Power

The quantity

$$\mathbf{W} = \frac{1}{2}\mathbf{P}\mathbf{U_S}^*$$

is called the *vector power* supplied to the portion of the circuit to the right of the terminals in Fig. 22. Making use of Kirchoff's Pressure Law

$$\mathbf{P} = \mathbf{P_R} + \mathbf{P_M} + \mathbf{P_C}$$

$$\mathbf{P}\mathbf{U_S}^* = R_A \left|\mathbf{U_S}\right|^2 + j\omega M_A \left|\mathbf{U_S}\right|^2 - j\omega C_A \left|\mathbf{P_C}\right|^2 ,$$

so that

$$\mathbf{W} = \frac{1}{2}\mathbf{P}\mathbf{U_S}^* = \frac{1}{2}R_A \left|\mathbf{U_S}\right|^2 + j2\omega \left(\frac{1}{4}M_A \left|\mathbf{U_S}\right|^2 - \frac{1}{4}C_A \left|\mathbf{P_C}\right|^2\right) \tag{45}$$

Since $\frac{1}{2}R_A \left|\mathbf{U_S}\right|^2$ is the average power dissipated in the acoustic resistance, $\frac{1}{4}M_A \left|\mathbf{U_S}\right|^2$ is the average energy stored in the acoustic mass $\langle E_M \rangle$, and $\frac{1}{4}C_A \left|\mathbf{P_C}\right|^2$ is the average energy stored in the acoustic compliance $\langle E_C \rangle$, one has

$$\mathbf{W} = P_{\text{av}} + j2\omega \left(\langle E_M \rangle - \langle E_C \rangle\right). \tag{46}$$

10 Arithmetic for Complex Numbers

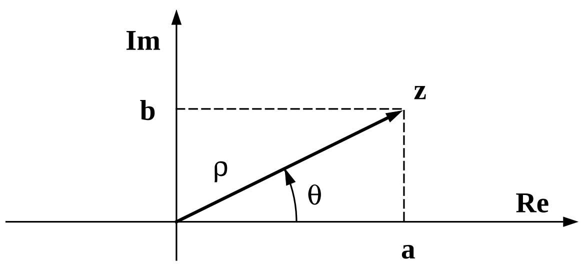

Figure 23: Representation of the complex number $\mathbf{z}$ as a point in the complex plane. The representation may be described in terms of rectangular components $\mathbf{z} = a + jb$, or equivalently in polar components $\mathbf{z} = \rho e^{j\theta}$.

In what follows, a, a_1, a_2, b, b_1, b_2, ρ, ρ_1, ρ_2 θ, θ_1, and θ_2 are real numbers, $\mathbf{z}$, $\mathbf{z_1}$, $\mathbf{z_2}$ are complex numbers, with

$$\mathbf{z} = \rho e^{j\theta} = a + jb \tag{47}$$

where

$$\begin{aligned} \rho &= |\mathbf{z}| = \sqrt{a^2 + b^2} && (48) \\ \theta &= \arctan b/a && (49) \\ \mathrm{Re}\,[\mathbf{z}] &= a = \rho \cos\theta && (50) \\ \mathrm{Im}\,[\mathbf{z}] &= b = \rho \sin\theta. && (51) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{z_1} &= \rho_1 e^{j\theta_1} = a_1 + jb_1 && (52) \\ \mathbf{z_2} &= \rho_2 e^{j\theta_2} = a_2 + jb_2. && (53) \end{aligned}$$

10.1 Addition and Subtraction

$$\begin{aligned} \mathbf{z_1} + \mathbf{z_2} &= (a_1 + a_2) + j(b_1 + b_2) && (54) \\ \mathbf{z_1} - \mathbf{z_2} &= (a_1 - a_2) + j(b_1 - b_2) && (55) \end{aligned}$$

10.2 Multiplication and Division

$$\mathbf{z_1} * \mathbf{z_2} = \rho_1 \rho_2 e^{j(\theta_1 + \theta_2)} = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1) \tag{56}$$

$$\frac{\mathbf{z_1}}{\mathbf{z_2}} = \frac{\rho_1}{\rho_2} e^{j(\theta_1 - \theta_2)} = \frac{(a_1 a_2 + b_1 b_2) + j(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2} \tag{57}$$

$$\frac{1}{\mathbf{z}} = \frac{1}{\rho} e^{-j(\theta)} = \frac{a - jb}{a^2 + b^2} \tag{58}$$

10.3 Complex Conjugates

The complex conjugate of $\mathbf{z}$, denoted by $\mathbf{z}^*$ has the following properties:

$$\mathbf{z}^* = \rho e^{-j\theta} = a - jb \tag{59}$$

$$\mathrm{Re}\,[\mathbf{z}^*] = \rho \cos\theta = a \tag{60}$$

$$\mathrm{Im}\,[\mathbf{z}^*] = -\rho \sin\theta = -b \tag{61}$$

$$\mathbf{z}\mathbf{z}^* = \rho^2 = a^2 + b^2 \tag{62}$$

$$\frac{\mathbf{z}}{\mathbf{z}^*} = e^{j2\theta} = \frac{(a^2 - b^2) + j(2ab)}{a^2 + b^2} \tag{63}$$

$$[\mathbf{z}^*]^* = \mathbf{z} \tag{64}$$

$$(\mathbf{z_1} \pm \mathbf{z}_2)^* = \mathbf{z_1}^* \pm \mathbf{z}_2^* \tag{65}$$

$$(\mathbf{z_1}\mathbf{z_2})^* = \mathbf{z_1}^*\mathbf{z_2}^* \tag{66}$$

$$\left(\frac{\mathbf{z_1}}{\mathbf{z_2}}\right)^* = \frac{\mathbf{z_1}^*}{\mathbf{z_2}^*} \tag{67}$$