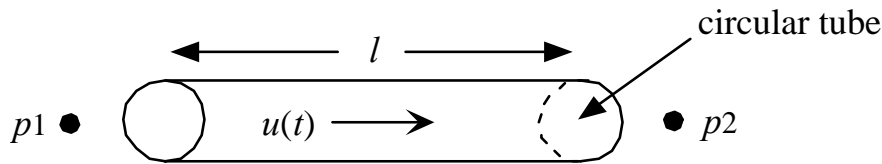


Lecture 10: Lumped Acoustic Elements and Acoustic Circuits

I. A Review of Some Acoustic Elements

A. An open-ended tube or Acoustic mass: units of kg/m⁴

linear dimensions l and $a < 0.1 \lambda$ and $S = \pi a^2$



$$p(t) = p1(t) - p2(t)$$

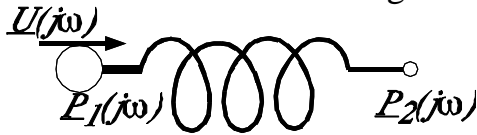
$$L_A = \frac{\rho_o l}{S} = \frac{\rho_o \text{Volume}}{S^2}$$

$$p(t) = L_A \frac{du(t)}{dt}$$

assumes only inertial forces

ρ_o = equilibrium mass density of medium

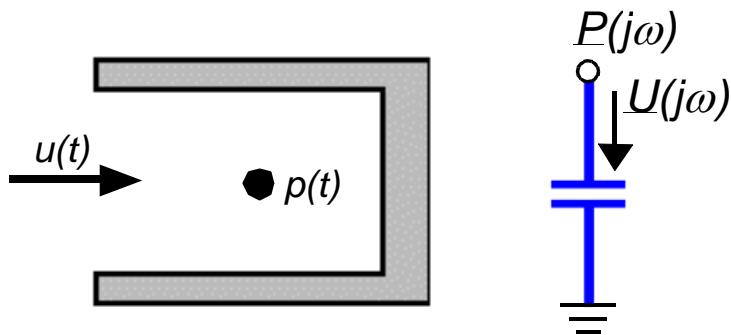
The Electrical Analog



$$P_1 - P_2 = U j\omega L_A$$

B. An Enclosed volume of air: units of Pa/m³.

linear dimensions $< 0.1 \lambda$

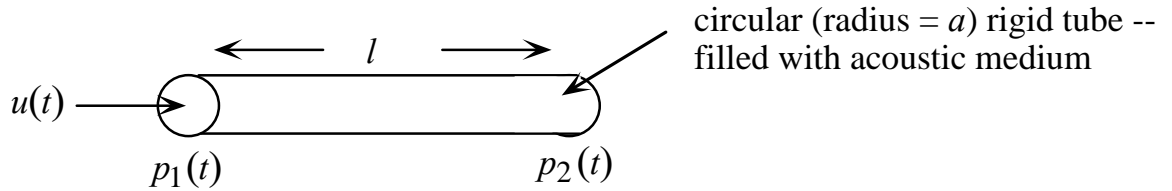


$$C_A = \frac{\text{Volume}}{\text{Adiabatic Bulk modulus}}$$

$$U = j\omega C_A P$$

C. A narrow tube or Acoustic Resistance: units of Acoustic Ohms ($\text{Pa}\cdot\text{s}/\text{m}^3$)

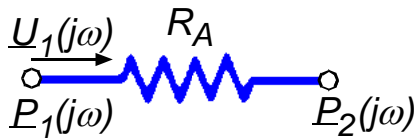
radius $a \ll 0.001 \lambda$



$$p(t) = p_1(t) - p_2(t) = R_A u(t) \leftarrow \text{assumption; only viscous forces}$$

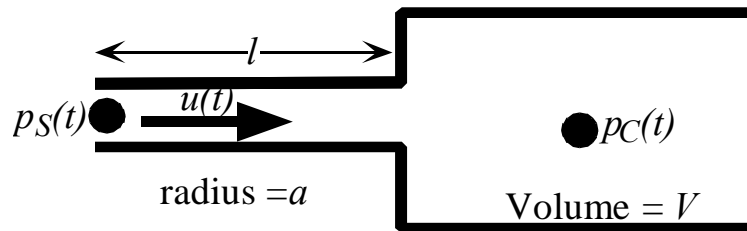
$$R_A = \frac{8\eta l}{\pi a^4} \Rightarrow \frac{p_1(t) - p_2(t)}{u(t)}$$

η = viscosity of medium

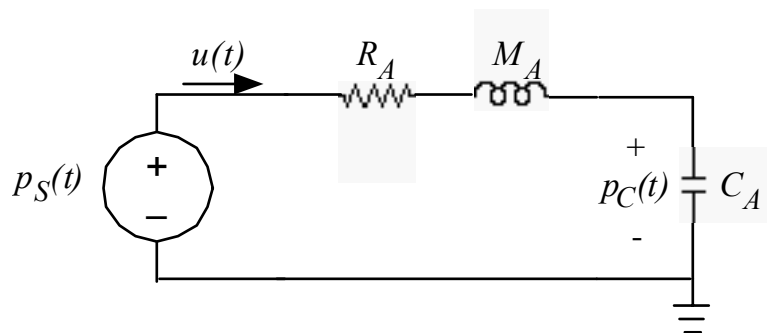


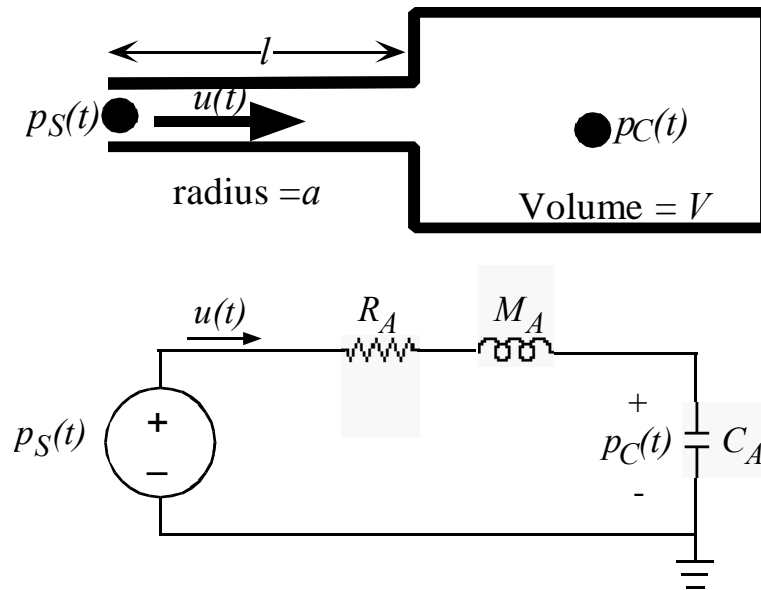
where $\underline{P}_1 - \underline{P}_2 = \underline{U}_1 R_A$.

II. An acoustic Circuit: The Helmholtz Resonator



Consider the case where the stimulus is a sound pressure source $p_1(t) = p_S(t)$. Since all of the volume velocity going into the lumped mass and resistance of the small tube goes into the cavity, a series arrangement of the elements is appropriate





Another way of expressing this series arrangement is that the impedance loading the source, the *input impedance* of the circuit is the sum of the impedances of the three elements:

$$Z_A^{IN}(s) = \frac{P(s)}{U(s)} = R_A(s) + sM_A + \frac{1}{sC_A}.$$

A. Relating the Different Acoustic Variables within a Lumped System: System functions

The two circuit representations are shorthand descriptions of differential equations that describe the relationship between system variables. In the case of pressure source the system function that describes the ratio of the volume-velocity at the input to the sound pressure of the input drive is known as the *Acoustic Input Admittance* Y_A . A second system function is the ratio of the sound pressure inside the jug to the stimulus sound pressure.

For the Helmholtz Resonator with a Sound Pressure source

$$\frac{U(s)}{P_S(s)} = Y_A^{IN}(s) = \frac{1}{R_A + sM_A + \frac{1}{sC_A}} = \frac{sC_A}{s^2 M_A C_A + sC_A R_A + 1}$$

$$\frac{P_C(s)}{P_S(s)} = \frac{\frac{1}{sC_A}}{R_A + sM_A + \frac{1}{sC_A}} = \frac{1}{s^2 M_A C_A + sC_A R_A + 1} \quad (\text{Eqn. 10.1A\&B})$$

if R_A Approximates 0

$$\frac{U(s)}{P_S(s)} \approx \frac{sC_A}{s^2 M_A C_A + 1}$$

$$\frac{P_C(s)}{P_S(s)} \approx \frac{1}{s^2 M_A C_A + 1}$$

Where are the poles and where are the zeros?

Writing 10.1 in terms of a sinusoidal drive yields

$$\frac{\underline{U}(\omega)}{\underline{P}_S(\omega)} = \underline{Y}^A(\omega) = \frac{1}{R_A + j\omega M_A + \frac{1}{j\omega C_A}} = \frac{j\omega C_A}{-\omega^2 M_A C_A + j\omega C_A R_A + 1}$$

$$\frac{\underline{P}_C(j\omega)}{\underline{P}_S(j\omega)} = \frac{\frac{1}{j\omega C_A}}{R_A + j\omega M_A + \frac{1}{j\omega C_A}} = \frac{1}{-\omega^2 M_A C_A + j\omega C_A R_A + 1} \quad (\text{Eqn. 10.2})$$

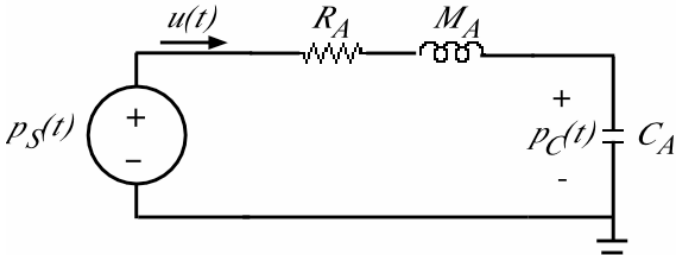
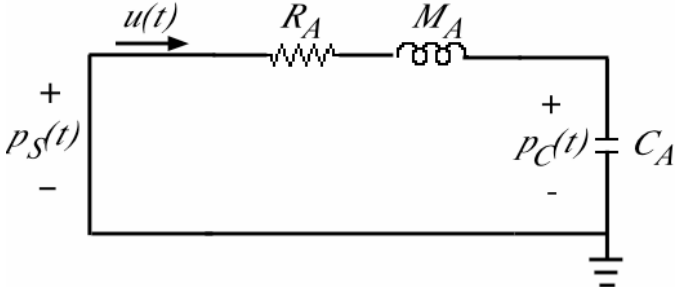
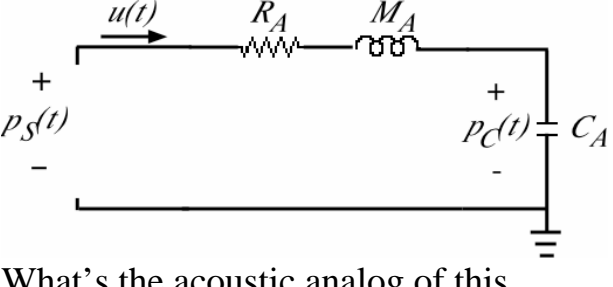
Again if R_A is small, the two system functions have poles at approximately

$$\omega_p \approx \pm \sqrt{\frac{1}{M_A C_A}}, \quad (\text{Eqn. 10.3})$$

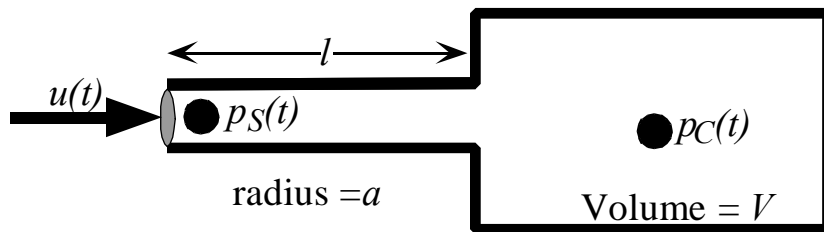
While the Admittance $\underline{Y}^A(\omega)$ (Eqn 10.2A) also has a zero at $\omega = 0$.

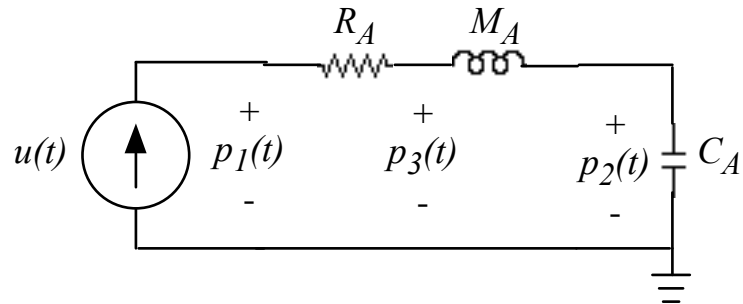
2. Natural Frequencies with a Sound Pressure Source.

Natural frequencies are frequencies where there will be some energy existing within the system when the input drive is turned off. These natural frequencies are essentially the poles of the transfer function. Equation 10.3 shows that the natural frequencies will depend on the dimensions of the bottle.

<p>Model with the source on.</p> 	<p>The <i>Natural frequencies</i> are those you see when the sources are turned off, which of the circuits in the next line is an appropriate model for turning the pressure source off?</p>
 <p>What's the acoustic analog of this "short circuit" ?</p>	 <p>What's the acoustic analog of this "open circuit"?</p>

3. A Helmholtz Resonator with a Volume-velocity source: System Functions and Natural Frequencies





System Functions for the bottle with a Volume Velocity source are:

$$\frac{P_1(s)}{U} = Z_A(s) = R_A + sM_A + \frac{1}{sC_A}$$

$$Z_A(s) = \frac{s^2 M_A C_A + s C_A R_A + 1}{s C_A} \quad (10.4A\&B)$$

$$\frac{P_2(s)}{U} = \frac{1}{s C_A}$$

in the sinusoidal steady state:

$$\frac{P_1(\omega)}{U} = Z_A(\omega) = \frac{-\omega^2 M_A C_A + j\omega C_A R_A + 1}{j\omega C_A} \quad (10.5A\&B)$$

$$\frac{P_2(\omega)}{U} = \frac{1}{j\omega C_A}$$

Again if R_A is near 0;

$$\frac{P_1(\omega)}{U} = Z_A(\omega) \approx \frac{-\omega^2 M_A C_A + 1}{j\omega C_A}$$

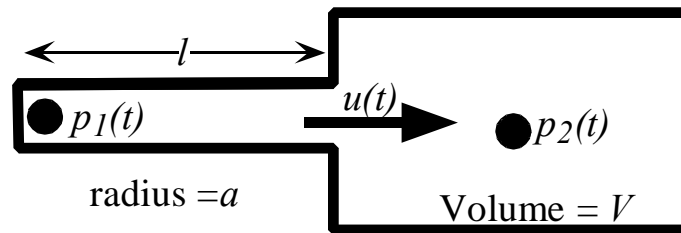
$$\frac{P_2(\omega)}{U} = \frac{1}{j\omega C_A}$$

Note that that the poles of 10.1 & 10.2 are the zeros of 10.4 & 10.5, and vice-versa, and that both 4.5A&B have the same pole.

What are the natural frequencies when the sound pressure source is turned off?

What is the condition of the bottle when the volume-velocity is turned off?

$u(t)=0$ corresponds to a rigid termination at the entrance of the bottle



What does the electric circuit look like, when the current source is turned off?

4. Frequency Dependence of the Impedances and Transfer Functions:

Equation 10.5A defines how the impedance magnitude and angle of varies with frequency ω .

Case 1: In the low-frequency limit, when the frequency of the stimulus is very small:

10.5A can be approximated by

$$\frac{P_1(\omega)}{U(\omega)} = \frac{1}{j\omega C_A} \quad (10.6)$$

In the low-frequency limit: the impedance is imaginary with a magnitude that is inversely proportional with frequency and an angle of $-\pi/2$.

Case 2: When the frequency is very large. The numerator is dominated by $\omega^2 M_A C_A$ and 10.5A can be approximated by

$$\frac{P_1(\omega)}{U(\omega)} = j\omega M_A, \quad (10.7)$$

such that the magnitude is proportional to frequency and the angle is $+\pi/2$.

Case 3: With R small, Eqn, 10.5A has a zero at some Middle Frequency

$$\omega_0 \approx 1/\sqrt{C_A M_A}, \quad (10.8)$$

This behavior indicates a zero in the impedance. What is the angle of the impedance at this '*resonance frequency*'

5 Bode Plots of Complex Amplitudes in the Sinusoidal Steady State

Equation 10.5A demonstrates how the input impedance functions vary with ω .

For a model coke bottle:

$$\text{Vol} = 0.5 \text{ liter} = 0.5 \times 10^{-3} \text{ m}^3$$

$$l = 5 \text{ cm} = 0.05 \text{ m}; a = 0.005 \text{ cm}$$

$$C_A = \text{Vol} / (1.4 \times 10^5) = 3.6 \times 10^{-9} \text{ Pa/m}^3$$

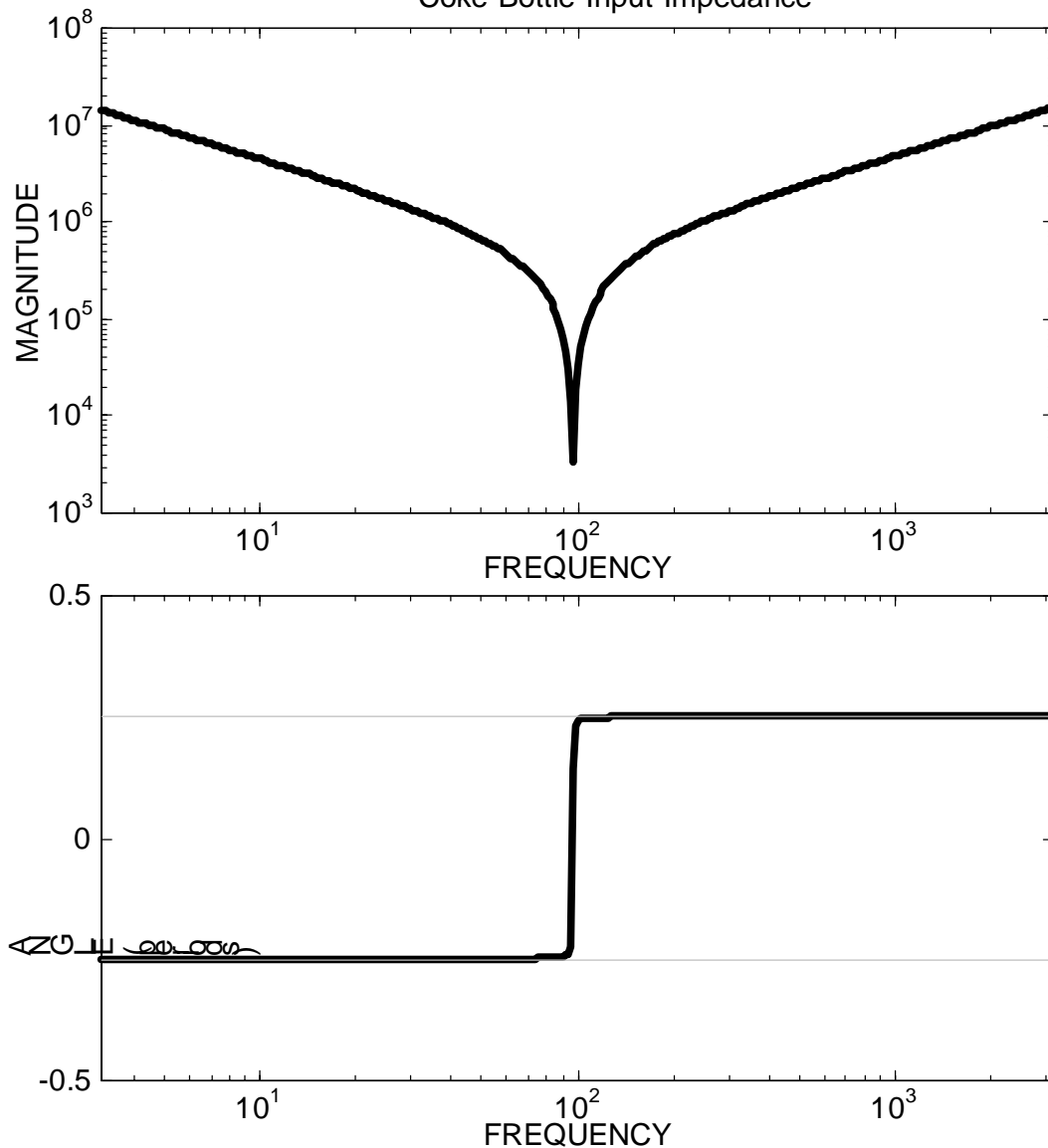
$$M_A = 1.18 * 0.05 / (\pi a^2) = 750 \text{ kg/m}^4$$

$$R_A \approx 1000 \text{ Pa-s/m}^3;$$

$$Z_{INPUT}^A = R_A + j\omega M^A + \frac{1}{j\omega C^A}$$

INPUT IMPEDANCE

Coke Bottle Input Impedance



6. Input Impedance and Power

The input impedance of a system is an indicator of power absorption and power storage within the system. The rate of energy absorption by an acoustic system can be defined by the average power at the input:

$$W_{AV} = \frac{1}{2} \operatorname{Re} \{ \underline{P}_{IN} \underline{U}_{IN}^* \} = \frac{1}{2} | \underline{P}_{IN} | | \underline{U}_{IN}^* | \cos(\angle \underline{P}_{IN} - \angle \underline{U}_{IN}). \quad (10.9)$$

Note that the argument to the cosine function on the right side of (10.9) is the angle of the impedance. The cosine term is maximum (with a value of 1) when the impedance angle is 0. If the impedance is dominated by some reactive term (impedances that are dominated by imaginary terms) such that $\angle \underline{Z}_{IN} \approx \pm\pi/2$, then the average power approximates 0 and little power is absorbed.

In the case of reactive impedances, little power is absorbed by the system, instead the system is periodically storing sound energy and returning it to the source on a cycle-by-cycle basis. In the case of an acoustic compliance, the energy stored W_C (with units of joules) stored in the compliance at any one time is proportional to the square of the pressure difference across the compliance, $p_C(t)$, such that the potential energy stored in the compliance is maximum at \pm peak pressures and 0 when the sound pressure = 0:

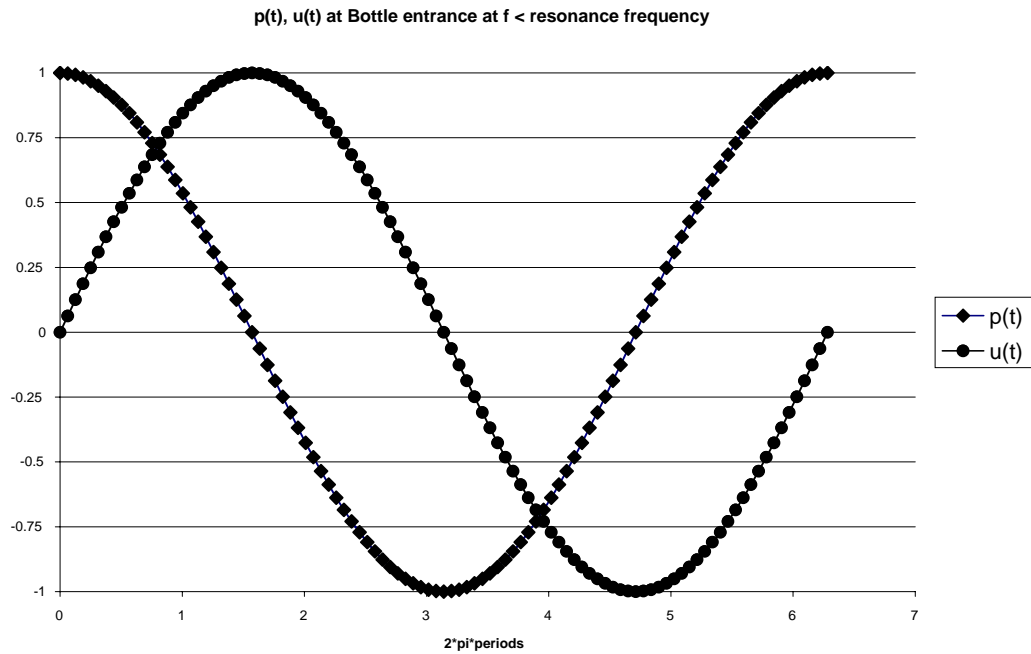
$$E_C(t) = \frac{1}{2} C_A p_C^2(t) . \quad (10.10)$$

In the case of an acoustic inertance, kinetic energy is alternately stored and returned in the momentum of the air particles and is largest during times of peak velocity:

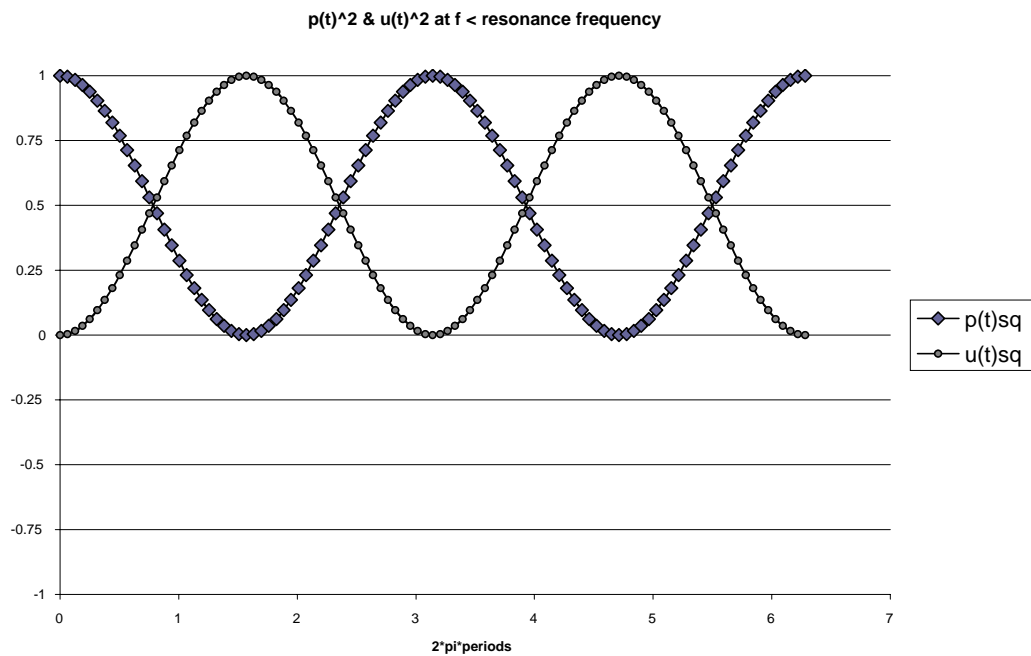
$$E_M(t) = \frac{1}{2} M_A u^2(t) . \quad (10.11)$$

The sign of the reactance at any frequency, negative for compliance dominated reactances and positive for inertance dominated reactances, tells you how energy is stored in the system at that frequency. Input impedance zeros occur when there is a balance between the energy stored and released by a compliance and a mass.

A sketch of the normalized pressure and volume velocity at the entrance to the bottle at some frequency below the resonant or natural frequency.



A sketch of the normalized pressure and velocity squared at the entrance to the bottle at some frequency below the resonant or natural frequency.



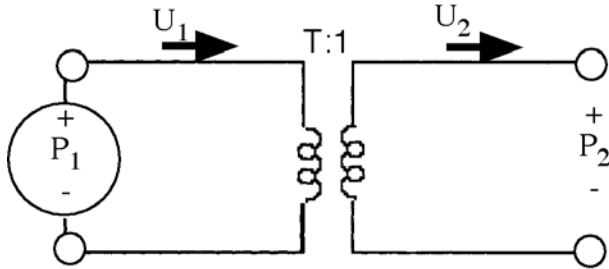
What do these curves look like above the resonant frequency?

What do we know about the magnitude of the stored potential and kinetic energy at the resonant frequency?

II. Mechano-Acoustic Transformers & Transducers

1. Ideal Transformers and Transformation of variables

a. One example of a two-port is an Ideal Electrical transformer, where:



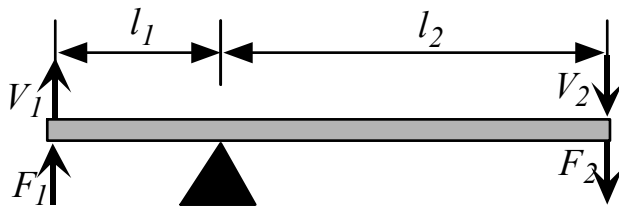
$$\frac{U_2}{U_1} = T = \frac{P_1}{P_2}$$

with $T = \text{“the turns ratio”}$

How does the transformer ‘Transform’ impedance?

Note that the Transformer turns ratio T , in this case, is dimensionless.

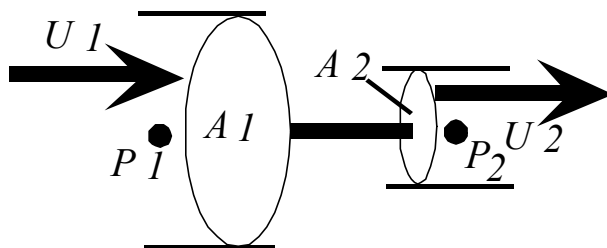
b. An example of an ideal mechanical transformer is a massless rigid lever arm..



$$\frac{V_2}{V_1} = T = \frac{F_1}{F_2}$$

where $T = l_2/l_1$
and is dimensionless.

c. Ideal coupled pistons act as (dimensionless) acoustic transformers.

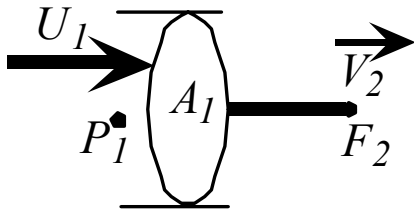


$$\frac{U_2}{U_1} = T = \frac{P_1}{P_2}$$

where $T = A_2/A_1$
and is dimensionless.

How does the piston ‘Transform’ acoustic impedance?

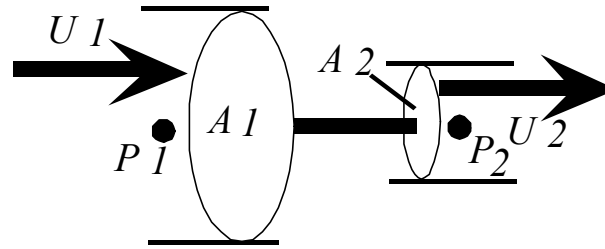
d. An ideal piston and a coupled rod can act as an acoustico-mechanical transformer



$$\frac{V_2}{U_1} = T = \frac{P_1}{F_2}$$

where $T = l/A_1$ with units of 1/area

e. A first order model of the vertebrate middle ear is as an acoustic transformer that couples acoustic power from air to the fluid-filled inner ear.



$$\frac{U_2}{U_1} = T = \frac{P_1}{P_2}$$

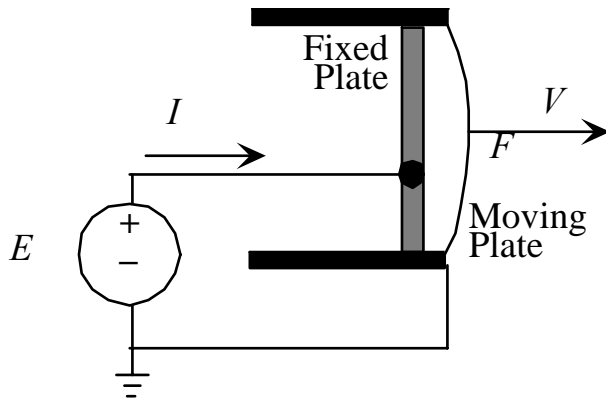
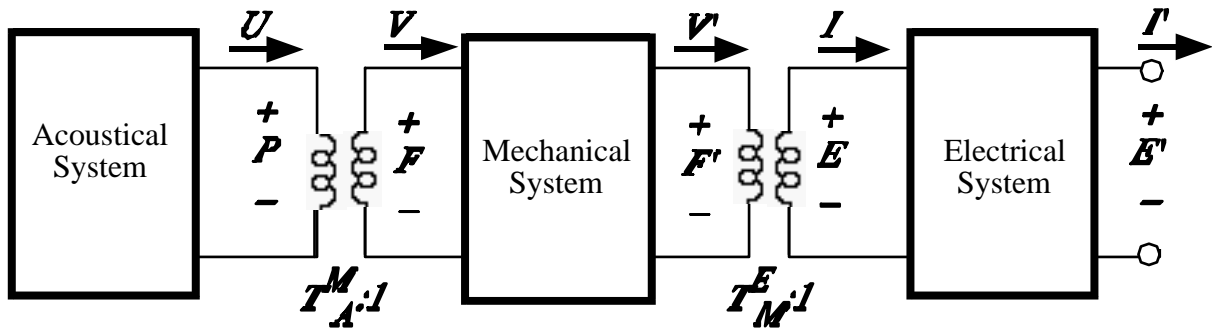
<p>A cross-section through half a lizard head</p>	<p>In the lizards the tympanic membrane is about 20 times larger in area than that of the boney footplate of the ossicle (the wide part that couples to one of the inner ear windows). This leads to a transformer ratio of 1/20 that would suggest.</p> $\frac{U_2}{U_1} = \frac{VA_{FP}}{VA_{TM}} \approx \frac{1}{20}$ <p>The Impedance transformation ratio is $T^2=1/400$:</p> <p>If $\frac{P_2}{U_2} = 400$; $\frac{P_1}{U_1} = 1$</p>
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The acoustic transformer trades a change in volume velocity for a change in pressure of opposite magnitude. As we will see in the next two lectures, the ideal transformer is an idealization in the real world its difficult to build a massless, rigid piston that is perfectly mobile.

f. An acoustic – mechanical – electric transducer : The capacitive microphone

The signal flow through a capacitive microphone can be separated into three separate stages:

- An Acoustic stage that generates the acoustic input signal:
- A Mechanical stage that gathers the force produced by the acoustic signal
- An electric system that transforms mechanical force and motion into voltage and current



mechanical transformer such that:

The heart of an *electro-static* transducer is a charged capacitor with a fixed internal plate and a moving external plate where the capacitance depends on the distance between the plates and the distance is a function of the voltage across the plates.

Such a transducer acts as an *electro -*

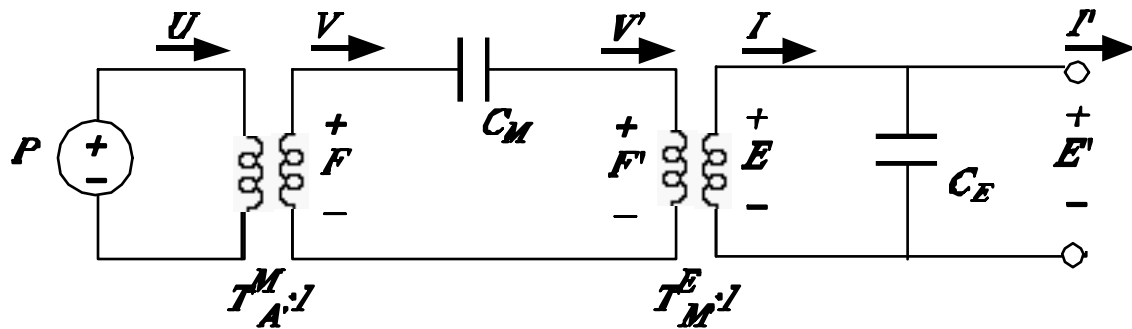
$$\frac{V}{I} = T_{ES} = \frac{E}{F}$$

$$\text{where } T_{ES} = \frac{x_0(\text{the static separation of the plates})}{C_0(\text{static capacitance})E_0(\text{static voltage})}$$

when the voltage induced variations in x , C and F are small. Input voltages produce a force on and velocity of the moving plate (or diaphragm) that when integrated over the surface of the moving plate produce a volume velocity and a sound pressure.

f. A Low-Frequency Model of the Electrostatic Microphone:

In the electrostatic microphone, the microphone acts simply as long as the capacitance of the diaphragm and backplate C_E controls the electrical stage and the compliance of the diaphragm C_M controls the mechanical stage.



In our circuit, C_M is placed in series with the acoustic source, since the force and motion is gathered by motion of the diaphragm.

C_E is placed in parallel because it represents the static electrical capacitance which is there even when the diaphragm is fixed and V and $I = 0$.

Expensive ‘capacitive’ microphones with small (high-impedance) capacitance and stiff (low compliance) membranes are used as “STANDARD MICROPHONES” throughout the world.