Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.551J / HST 712J
Notes on Acoustic Circuits
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LECTURE 8

5 & 7 October 2004

1 The Concept of Finite (Lumped) Elements

Although sound can always be thought of as a wave phenomenon, it is often not possible to solve analytically the partial differential equations that govern sound propagation. This difficulty generally arises when the structure of the boundary conditions is complex, as in most physiological and many mechanical systems.

In certain restricted conditions, it is possible to develop approximations that allow the sound field to be determined with high accuracy without resorting to the partial differential equation formalism. These approximations rely on the fact that over sufficiently short distances sound (pressure, $p(x,t)$, and particle velocity, $u(x,t)$) can be treated as varying linearly with position over the short distances:

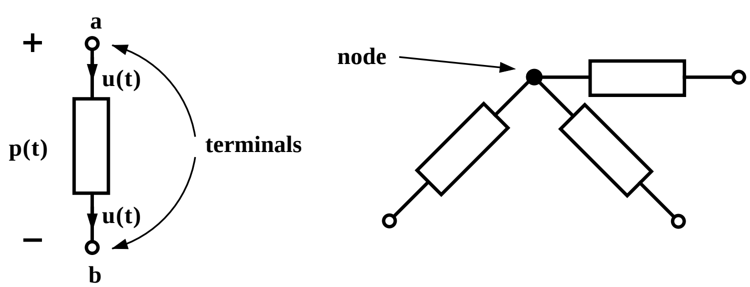

Figure 1: General acoustic circuit element, illustrating the terminals of the element, the pressure p across the element, the volume velocity u through the element, and the node at the connection of two terminals from different elements.

2 Acoustic Circuits

Acoustic circuits consist of elements that interact with each other at distinct terminals (Fig. 1). Each element has a pressure difference $p(t)$ *across* its and a volume velocity $u(t)$ *through* it, entering the element at one terminal and leaving at the other. The pressure difference $p(t)$ corresponds to the difference between the acoustic pressures at different points in space. The volume velocity $u(t)$ is the product of the particle velocity and the area characteristic of the physical element. Use of volume velocity, rather than particle velocity, simplifies the formulation of the conservation equations (Sec. 4) for acoustic circuits. The rate at which work is done on an element by the rest of the circuit to which the element is connected is the acoustic power $w(t) = p(t)u(t)$.

3 Acoustic Elements

This section introduces three fundamental acoustic elements: the acoustic mass, the acoustic compliance, and the acoustic resistance, and two acoustic sources: volume velocity source, and the pressure source.

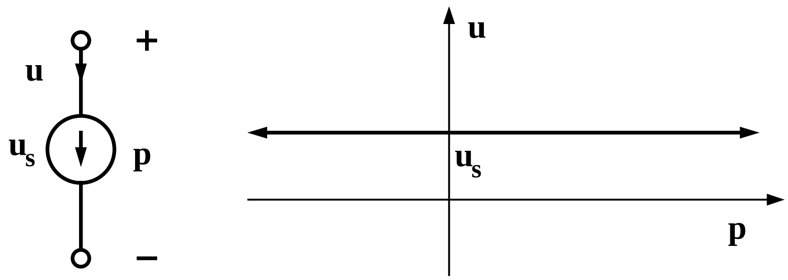

Figure 2: The circuit representation and terminal characteristic of a volume velocity source with intrinsic volume velocity u_S.

3.1 Volume Velocity Source

A volume velocity source (Fig. 2) moves a specified volume velocity from one point in an acoustic circuit to another independent of the pressure difference between the two points. The circuit symbol for and terminal characteristics of a volume velocity source are shown in Fig. 2. The pressure p across the terminals of a volume velocity source is determined by the intrinsic volume velocity of the source u_S and the rest of the network to which the volume velocity source is connected. Note that the rate at which work is done on a volume velocity source, $w = pu = pu_S$ may be positive or negative, so that the volume velocity source may do work on the rest of the network or have work done on it.

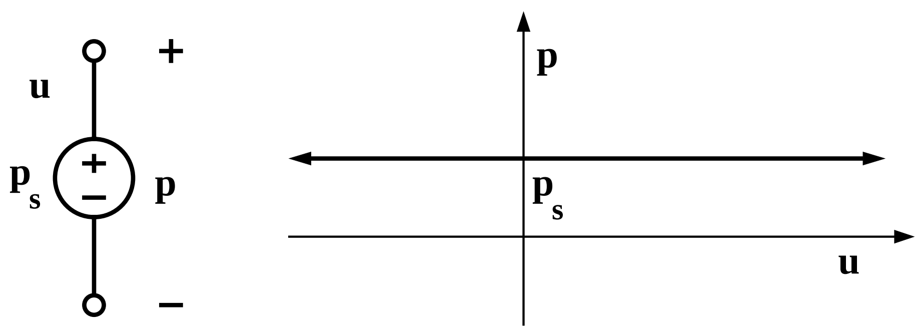

Figure 3: The circuit representation and terminal characteristic of a pressure source with intrinsic pressure p_S.

3.2 Pressure Source

A pressure source (Fig. 3) imposes a specified pressure difference between two points in an acoustic circuit independent of the volume velocity through the source. The circuit symbol for and terminal characteristics of a pressure source are shown in Fig. 3. The volume velocity u that flows through a pressure source is determined by the intrinsic pressure of the source p_S and the rest of the network to which the pressure source is connected. Note that the rate at which work is done on a pressure source, $w = pu = p_S u$ may be positive or negative, so that the pressure source may do work on the rest of the network or have work done on it.

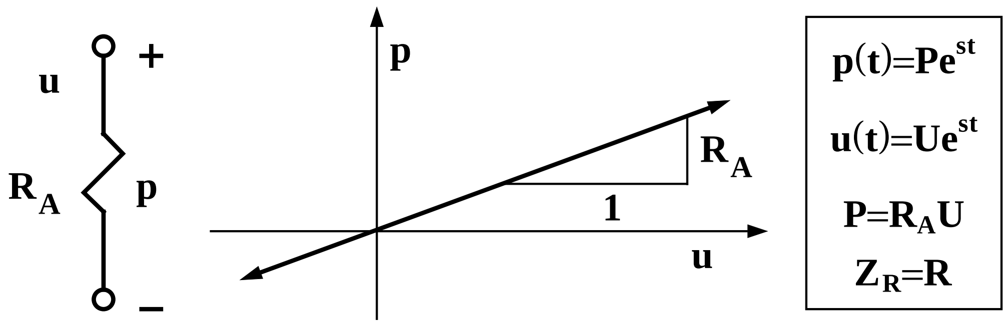

Figure 4: The circuit representation and terminal characteristic of an acoustic resistance with resistance R_A.

3.3 Acoustic Resistance

An acoustic resistance (Fig. 4) develops a pressure difference across its terminals that is proportional to the volume velocity through the resistance. The circuit symbol for and terminal characteristics of an acoustic resistance are shown in Fig. 4. The pressure p across the terminals of an acoustic resistance is proportional to volume velocity that flows through the resistance R_A, $p = R_A u$. The rate at which work is done on an acoustic resistance is $w = pu = R_A u^2 = p^2/R_A$. For a *physical* acoustic resistance, $R_A \geq 0$, so $w \geq 0$, i.e. an acoustic resistance cannot do work on the rest of the network to which it is connected.

If the volume velocity through an acoustic resistance is $u(t)$ the pressure across the terminals of the resistance is $p(t) = R_A u(t)$. Thus the waveforms $u(t)$ and $p(t)$ have the same shape, differing only in scale factor (including units) In the special case

$$u(t) = Ue^{st} \tag{1}$$

$$p(t) = Pe^{st} = R_A U e^{st}. \tag{2}$$

so that $P = R_A U$. $Z_R = P/U = R_A$ is said to be the acoustic *impedance* of the acoustic resistance.

3.4 Acoustic Mass

An acoustic mass (Fig. 5) develops a pressure difference across its terminals that is proportional to the time rate of change of volume velocity through the mass. The circuit symbol for and terminal characteristics of an acoustic mass are shown in Fig. 5. The pressure p across the terminals of an acoustic mass is proportional to the time rate of change of the volume

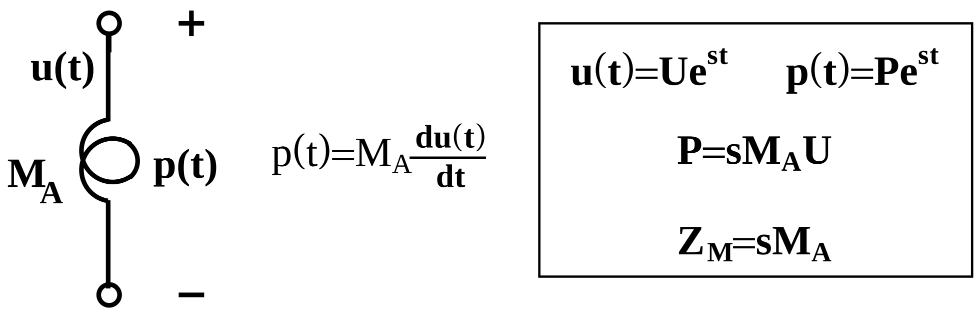

Figure 5: The circuit representation and terminal characteristic of an acoustic mass with mass M_A.

velocity that flows through the acoustic mass M_A, $p(t) = M_A \frac{du}{dt}$. The rate at which work is done on an acoustic mass is

$$w(t) = p(t)u(t) = M_A \frac{du(t)}{dt} u(t) = \frac{dE_M(t)}{dt},$$

where

$$E_M(t) = \frac{1}{2} M_A u^2(t)$$

is the energy stored in the acoustic mass. When $w(t) > 0$, the rest of the circuit is causing the energy stored in the acoustic mass to increase.

If the volume velocity through an acoustic mass is $u(t) = Ue^{st}$, the pressure across the terminals of the mass is

$$p(t) = M_A \frac{du(t)}{dt} = sM_A Ue^{st},$$

so that $p(t) = Pe^{st}$, where $P = sM_A U$. $Z_M = P/U = sM_A$ is said to be the acoustic impedance of the acoustic mass. Note that unlike the impedance of an acoustic resistance, the impedance of an acoustic mass is dependent on the value of the generalized frequency parameter s.

3.5 Acoustic Compliance

The volume velocity through an acoustic compliance (Fig. 6) is proportional to the time rate of change of acoustic pressure across the compliance. The circuit symbol for and terminal characteristics of an acoustic compliance are shown in Fig. 6. The volume velocity u through an acoustic compliance is proportional to the time rate of change of the pressure across the terminals of the acoustic compliance C_A, $u(t) = C_A \frac{dp}{dt}$. The rate at which work is done on an acoustic compliance is

$$w(t) = p(t)u(t) = p(t)\, C_A \frac{dp(t)}{dt} = \frac{dE_C(t)}{dt},$$

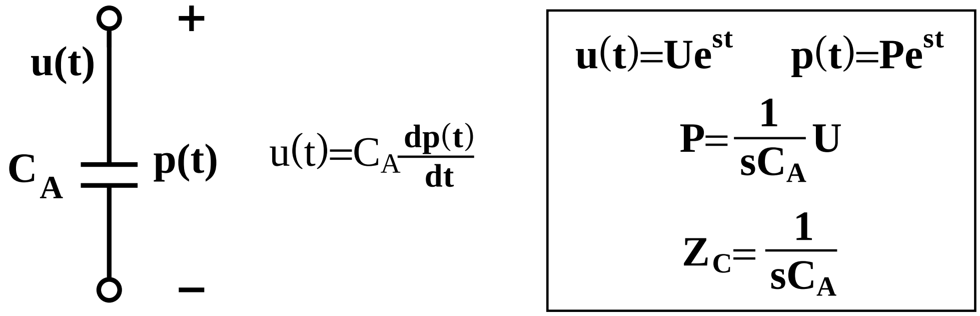

Figure 6: The circuit representation and terminal characteristic of an acoustic compliance with compliance C_A.

where

$$E_C(t) = \frac{1}{2} C_A\, p^2(t)$$

is the energy stored in the acoustic compliance. When $w(t) > 0$, the rest of the circuit is causing the energy stored in the acoustic compliance to increase.

If the pressure across the terminals of an acoustic compliance is $p(t) = Pe^{st}$, the volume velocity through the compliance is

$$u(t) = C_A \frac{dp(t)}{dt} = sC_A\, Pe^{st},$$

so that $u(t) = Ue^{st}$ where $U = sC_A\, P$. $Z_C = P/U = 1/sC_A$ is said to be the acoustic impedance of the acoustic compliance. Note that unlike the impedance of an acoustic resistance, the impedance of an acoustic compliance is dependent on the value of the generalized frequency parameter s.

4 Connections of Acoustic Elements

An acoustic circuit is an interconnection of acoustic elements. The connections occur at the terminals of the elements, which become the *nodes* of the circuit. The connections allow volume velocity and pressure to be shared among elements. The precise consequences of making these connections were first stated as laws (in the context of electrical circuits) by Kirchoff in the nineteenth century.

4.1 KUL - Kirchoff's Volume Velocity Law

Kirchoff's Volume Velocity Law expresses the conservation of mass in circuit terms. For each node in an acoustic circuit, e.g., Fig. 7, the algebraic sum of the volume velocities entering (or leaving) the node must be zero, otherwise mass would accumulate at the node. If there are N nodes in the circuit, N equations expressing KUL can be written, but only $N - 1$ of

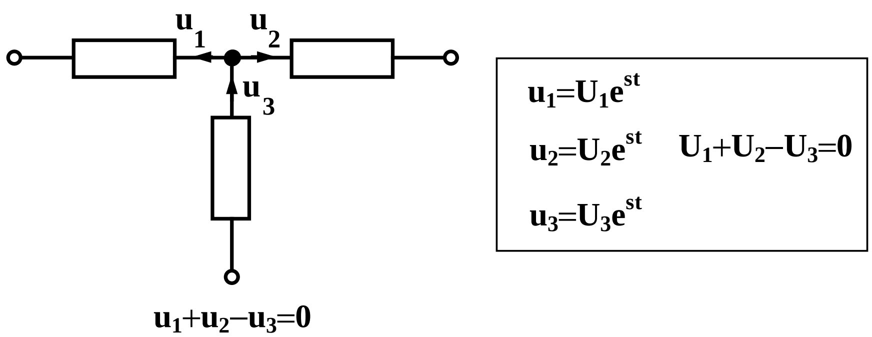

Figure 7: An example of the application of Kirchoff's Volume Velocity Law. For volume velocities with arbitrary time dependence, KUL applies to the values of u_1, u_2 and u_3 at each instant of time, $u_1 + u_2 - u_3 = 0$. If all volume velocities have the same e^{st} time dependence, KUL is satisfied at each instant of time if it is satisfied by the values of the volume velocities at $t = 0$, U_1, U_2, and U_3, so that $U_1 + U_2 - U_3 = 0$.

these are linearly independent. As a result, KUL is satisfied automatically at the Nth node if it is satisfied at the other $(N - 1)$ nodes.

4.2 KPL - Kirchoff's Pressure Law

Kirchoff's Pressure Law expresses the conservation of work in circuit terms. For each closed loop in an acoustic circuit, e.g., Fig. 8, the algebraic sum of the pressures encountered in a traverse of the loop node must be zero, otherwise net work would be done during the traverse.

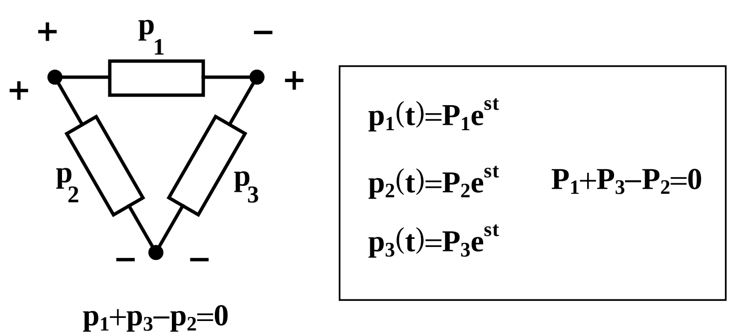

Figure 8: An example of the application of Kirchoff's Pressure Law. For pressures with arbitrary time dependence, KPL applies to the values of p_1, p_2, and p_3 at each instant of time, $p_1 + p_3 - p_2 = 0$. If all pressures have the same e^{st} time dependence, KPL is satisfied at each instant of time if it is satisfied by the values of the pressures at $t = 0$, P_1, P_2, and P_3 so that $P_1 + P_3 - P_2 = 0$.

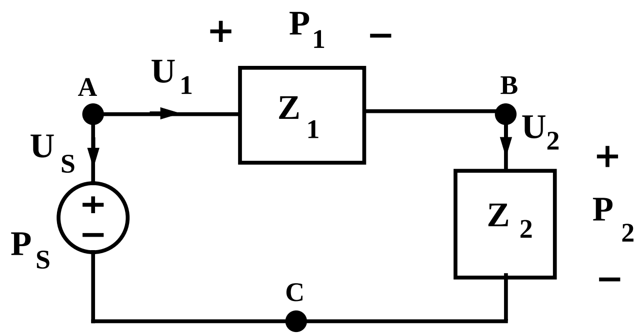

Figure 9: Example of an acoustic circuit consisting of a pressure source and two elements connected *in series*. All pressures and volume velocities are assumed to have the same e^{st} time dependence.

5 Example - Elements in Series

In the circuit of Fig. 9, KUL may be expressed at nodes A and B as

$$\begin{aligned} U_S + U_1 &= 0 \qquad \text{Node A.} \\ -U_1 + U_2 &= 0 \qquad \text{Node B.} \end{aligned}$$

Adding these two equations yields the KUL equation at Node C[1]

$$U_S + U_2 = 0.$$

The above equations indicate that

$$-U_S = U_1 = U_2 = U. \tag{3}$$

Connections of elements, such as that in Fig. 9 that require that two or more elements share a common volume velocity (ignoring sign) are said to be *series* connections.

In the circuit of Fig. 9, KPL may be expressed by traversing the single loop

$$-P_S + P_1 + P_2 = 0$$

or, equivalently,

$$P_S = P_1 + P_2. \tag{4}$$

Recognizing that $P_1 = U_1 Z_1 = U Z_1$ and that $P_2 = U Z_2$, one has

$$P_S = U(Z_1 + Z_2). \tag{5}$$

[1]This confirms, as noted above, that the KUL equation at the third node is linearly dependent on the KUL equations at the other two nodes.

Provided $Z_1 + Z_2 \neq 0^2$, U can be determined by dividing both sides of this equation by $Z_1 + Z_2$:

$$U = \frac{1}{Z_1 + Z_2} P_S \tag{6}$$

This result indicates that the series connection of impedances Z_1 and Z_2 is equivalent to a single impedance of value $Z_1 + Z_2$.

Making use of Eq. 6, pressures P_1 and P_2 are readily determined

$$P_1 = \frac{Z_1}{Z_1 + Z_2} P_S \tag{7}$$

$$P_2 = \frac{Z_2}{Z_1 + Z_2} P_S. \tag{8}$$

Equations Eq. 7 and Eq. 8 indicate that P_S, the pressure across nodes A and C, divides across the two series-connected impedances in proportion to the value of the impedance, i.e.,

$$\frac{P_1}{P_2} = \frac{Z_1}{Z_2}$$

In contrast to the series connection, in which elements share a common volume velocity, elements share a common pressure when connected in parallel (Problem Set 3).

For both series and parallel connections of impedances across a pair of terminals, there is an equivalent impedance that is indistinguishable when connected across the same terminals. This concept is generalized in Sec. 6.