6.6220 HW0 Solution*

0.1

Key take-aways

- Switches and diodes are non-linear devices, and the circuits they are part of are time varying, so we **can't** directly apply our Linear Time-Invariant "toolbox" to circuits that contain them. Instead, we decompose these "switching circuits" into stages, where each stage is modeled as an LTI circuit that we know how to solve.

- In order to know when uncontrolled switches like diodes turn on or off, we can use the method of assumed states (MAS). Don't forget that when voltages or currents in the circuit change polarity, the state of the diodes in the circuit can change. Eventually, you can rely on your intuition to quickly assess the state of a diode based on the voltage it would have to block or the current it would have to conduct.

- RLC circuits are common in switching circuits, and you should be comfortable solving these kinds of circuits in the time domain.

- In some instances, you can arrive at a solution quicker by using an energy balance perspective rather than a time-domain perspective.

(a) (1) At $t = 0$, the switch is closed. What is the state of the diode? Use the method of assumed states:

If the diode is on, then the voltage across R at $t = 0$ is V_x. This would mean that there is a current through the resistor which flows into the cathode of the diode — this is not possible! Therefore, the diode must be off at $t = 0$. An ideal diode is an open circuit when it's off, so our resulting circuit is an LC circuit. Write out KVL and KCL, get a (second-order) differential equation, and solve it. The result is of the form

$$V_c(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$
$$i_{coil}(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$. Our initial conditions are $V_c(0) = V_x$, $i_{coil}(0) = 0$, and $v_{coil}(0) = V_x$ (where we use the convention that a passive component has a voltage polarity such that current flows into its positive terminal). Thus,

$$V_c(t) = V_x\cos(\omega_0 t)$$
$$i_{coil}(t) = \frac{V_x}{\omega_0 L}\sin(\omega_0 t)$$

We're not done!

We arrived at this LC circuit by assuming that the diode was off, which was the case for the capacitor voltage being positive. However, our solution shows that the capacitor voltage is sinusoidal and will become zero at $t_1 = \pi/(2\omega_0)$. When this happens, the diode turns on (you can use the MAS here to confirm that) and the system becomes a parallel RLC circuit with the following state equation:

$$LC\frac{d^2 i_{coil}}{dt} + \frac{L}{R}\frac{di_{coil}}{dt} + i_{coil} = 0$$

Recall from differential equations that we solve this with a "characteristic equation":

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

which will have two roots.

We need numerical answers for these right now to determine the structure of the solution; plugging in, we get $s_1 = -8954$, $s_2 = -56406$. Since both of these roots are real, the solution will consist of exponentials:

$$i_{coil} = Ee^{s_1(t-t_1)} + Fe^{s_2(t-t_1)}$$

*Updated: Mansi Joisher 2023 (adapted from Yang 2022, Boles 2021, Ranjram 2020, Zhang 2019, Hanson 2018, and Yiou 2017) where E and F are coefficients to be determined. We also could have known that this was the solution by calculating the parallel RLC quality factor $Q = R\sqrt{C/L} = 0.34$ — since the Q is less than 1/2, the system is overdamped (exponential solutions).

Note that even though we "switched" into an RLC circuit from the original LC circuit at the instant $t = t_1$, we know that the voltage on a capacitor and the current through an inductor cannot change instantly. Thus, at $t = t_1$:

$$i_{coil}(t_1) = E + F = \frac{V_x}{\omega_0 L}$$
$$V_c(t_1) = L\frac{di_{coil}}{dt} = L(Es_1 + Fs_2) = 0$$

Solving for E and F gives us

$$E = +4.808V_x$$
$$F = -0.763V_x$$

The system is overdamped so it does not oscillate. Since the capacitor voltage at $t = t_1$ is zero, it cannot become zero again (the exponential solution will asymptotically approach zero as $t \to \infty$). Thus, the diode will remain on.

The final solution, then, is:

$$i_{coil} = \begin{cases} \frac{V_x}{L\omega_0}\sin(\omega_0 t) \approx 4.0452V_x \sin(22473.3t) & t \le t_1 \\ Ee^{s_1(t-t_d)} + Fe^{s_2(t-t_1)} \approx & \\ 4.808V_x e^{-8954(t-t_1)} - & t > t_1 \\ \quad 0.763V_x e^{-56406(t-t_1)} & \end{cases}$$

where t_1 is the time at which V_c falls below zero and can be found as $t_1 = \pi/(2\omega_0)$=69.9μs.

(2)

The coil current initially rises sinusoidally and reaches a maximum at t_1 after which it experiences an overdamped decay to zero. Thus, the maximum coil current occurs at $t = t_1$,

$$I_{max} = \frac{V_x}{\omega_0 L} = 3640.68 \text{ A} \quad (1)$$

Note that the solution to this problem can also be determined by conservation of energy. The inductor current reaches its maximum when the capacitor voltage is zero. At this point, all of the energy stored in the capacitor has been transferred to the capacitor. Thus, $\frac{1}{2}CV_x^2 = \frac{1}{2}LI_{max}^2$. Solving for I_{max} here yields the same solution as above.

(3)

As discussed in the solution for Part (1), the system oscillates for one quarter cycle before the diode turns on, so

$$t_1 = \frac{T}{4} = \frac{1}{4}\frac{2\pi}{\omega_0} = 69.9\ \mu\text{s}$$

(4)

The system begins with energy stored in the capacitor and ends in a state of no energy storage since the capacitor voltage and inductor current are zero. By conservation of energy, all of the initially stored energy must be dissipated in the resistor. Thus

$$E_{diss} = E_{stored,initial} = \frac{1}{2}CV_x^2 = 72.9 \text{ J}$$

c

We can repeat Part (1) with an initial capacitor voltage of $V_x = 450V$. The results are $I_{max} =$ 1820 A and $t_1 = 69.9\ \mu$s. Note that the timing of the diode turning on is *independent of* the initial voltage of the system. The current and voltage waveforms scale linearly with the initial conditions.

b

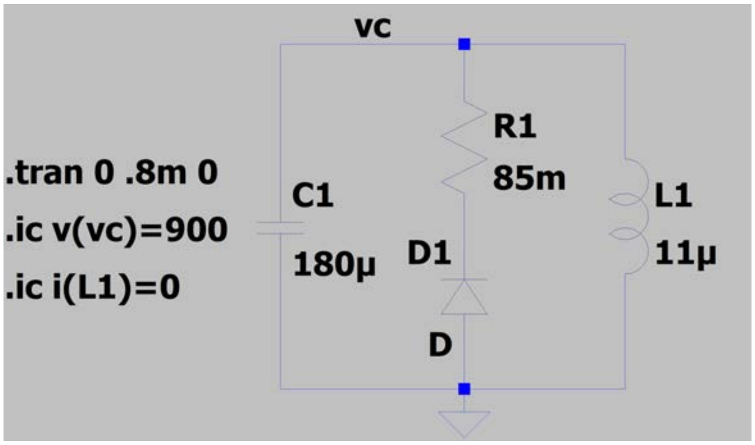

Figure 1: LTspice schematic, which corresponds to the circuit diagram for t>0.

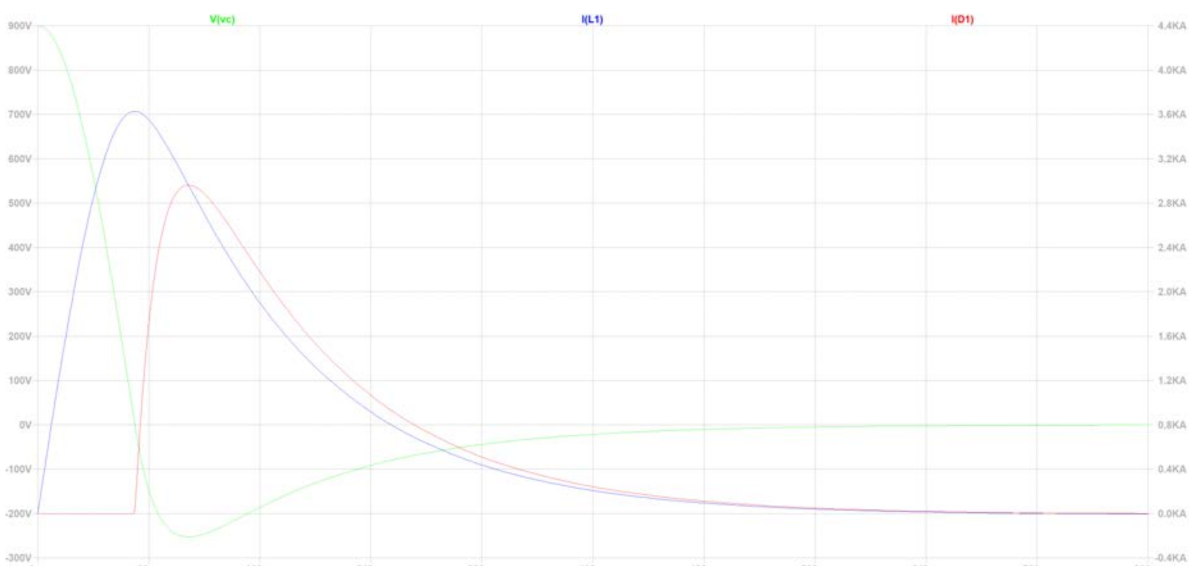

Figure 2: Simulated capacitor voltage, inductor current, and diode current waveforms (matching analytic solution).