6.6220 HW1 Solution*

1.1 Method of Assumed States

Refer to Fig. 3.3 in the text. Assume the diode stays on after t1. Immediately after t1, vd is negative. This negative voltage across the RL filter will act to create a negative current in the inductor. Since the inductor and diode are in series, a negative inductor current implies a negative diode current. This is inconsistent. The diode, therefore, must turn off at t1 and the source voltage is dropped across the diode over the interval $\omega t1 \leq \omega t \leq 2\pi$

1.2 ac Induction Motor

Note that this problem presents a relevant practical excitation and application. In the United States, most outlets are supplied at 120Vac, 60Hz. This is a root-mean-squared value, so the peak voltage is $120\sqrt{2} = 177$ Vac. Converting 60Hz to an angular frequency yields $2\pi 60 = 377 s^{-1}$.

Parts A&B

It's easiest to work in the frequency domain for this problem, using phasors and impedances.

The equivalent impedance seen by the voltage source is

$$\begin{aligned} Z &= R_s + j\omega L_{ls} + [j\omega L_m || (j\omega L_{lr} + R_r + R_x)] \\ &= R_s + j\omega L_{ls} + \frac{(j\omega L_m)(j\omega L_{lr}+R_r+R_x)}{j\omega(L_m+L_{lr})+R_r+R_x} \\ &= R_s + j\omega L_{ls} + \frac{-\omega^2 L_m L_{lr} + j\omega L_m (R_r+R_x)}{j\omega(L_m+L_{lr})+R_r+R_x} \end{aligned}$$

Plugging in $R_s = 0.08\Omega$, $L_{ls} = 1$ mH, $L_m = 40$ mH, $L_{lr} = 1$ mH, $R_r = 0.1\Omega$, $R_x = 33\Omega$, and $\omega = 377$ s^{-1}:

$$\begin{aligned} Z &= 5.72 + \mathrm{j}12.82 \\ &= 14.04\angle 65.96^\circ \end{aligned}$$

The current into the motor, i, is

$$\begin{aligned} i &= \tfrac{V}{Z} = \frac{170\angle 0}{14.04\angle 66^\circ} \\ &= 12.11\angle -66^\circ \\ &= 12.11\cos(377t - 1.151) \quad \mathrm{A} \end{aligned}$$

The equaton for power factor is

$$k_p = k_d k_\theta = (1)(\cos(1.151)) = 0.407 \text{ (lagging)}$$

where we note that because the voltage and current are purely sinusoidal, the distortion factor, k_d, is one. It is customary to use the sign convention that positive power factor represents an inductive load (current lags voltage), or to explicitly state whether the power factor is leading or lagging. This is important since the cosine in the definition of displacement factor, k_θ, removes information on the sign of the phase angle.

Part C

We computed that the utility interface is loaded inductively (positive reactance). A capacitor has negative reactance, so it is sensible to add one in order to improve the power factor seen by the utility. Assume we add a capacitor C in parallel with the motor. You can compute the new input impedance and determine the value of C that makes the reactive part of this impedance zero. Another way is to find an equivalent parallel circuit to represent the motor's impedance, Z, with a reactance jX and a resistance R. Once we do, we end up with a parallel RLC circuit, and we can just choose the capacitance that resonates with the positive reactance jX.

The overall impedance of such a parallel circuit is:

$$Z_{||} = \frac{jXR}{R+jX} = \frac{1}{R^2+X^2}\left(X^2 R + jXR^2\right)$$

*Updated: Mansi Joisher 2023 (adapted from Yang 2022, Boles 2021, Ranjram 2020, Zhang 2019, Hanson 2018, and Yiou 2017)

This needs to match Z, so

$$\frac{X^2 R}{R^2 + X^2} = 5.72$$
$$\frac{X R^2}{R^2 + X^2} = 12.82$$

Solving this set of equations gives us $X = 15.38\Omega$. Thus, we need a capacitor that can cancel this reactance:

$$\frac{1}{\omega C} = 15.38 \Rightarrow C = 172\mu F$$

1.3 Full Bridge Rectifier Load Regulation

Immediately in the problem description, we see that L is large enough to have small ripple. Thus, we can consider L to be a current source equal to I_D in our analysis (though this current source must still have an average voltage of zero like L).

3. A&B

Starting with the ideal case $L_c = 0$, when the input voltage is positive, the inductor pulls current through D1 and D2; when the input voltage is negative, the inductor pulls current through D3 and D4. With $L_c = 0$, the switching node v_x is always equal to V_s. The waveforms are sketched in Fig. 1.

The average voltage across L must be zero, and $\langle v_x \rangle = V_s$, so $V_D = V_s$. In that case, $I_L = I_D = V_s/R$.

3. C-F

Now the non-ideality L_c is added. When $v(t)$ goes from negative to positive, D3 and D4 stay on and D1 ,D2 must turn on to keep the output inductor fully supplied with I_D. With all four diodes on, v_x is at zero volts, and the current in L_c ramps up. Once i_c reaches $+I_L$, D3 and D4 can turn off. A similar case applies when the input voltage goes from positive to negative.

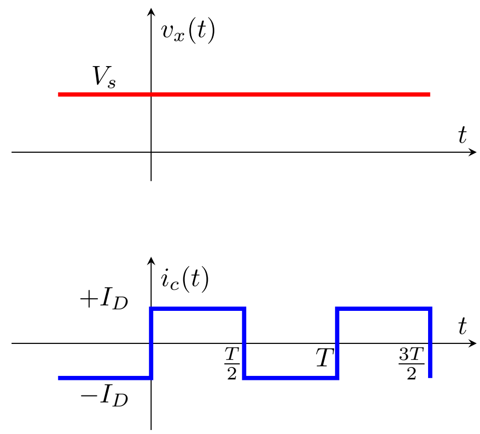

Figure 1: Switch node voltage and input current for the full bridge rectifier with no commutation

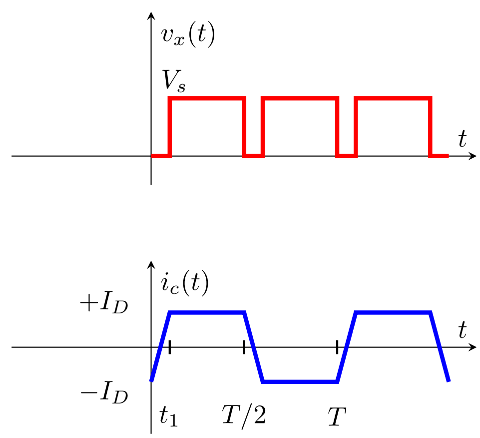

Figure 2: The effect of commutating inductance. There are now periods with all diodes on and $v_x = 0$ while the input voltage slews the commutating inductor current i_c.

The ramp-up and ramp-down time can be calculated as

$$t_1 = \frac{L_c(2I_D)}{V_s} \quad (1)$$

Note that I_D is not the same as it was before. Since v_x now equals 0 during commutation instead of being kept at V_s, $\langle v_x \rangle$ is lower, so $\langle V_D \rangle$ is lower, so

$\langle i_D \rangle$ is lower (for a fixed resistance). Indeed,

$$\langle V_D \rangle = \langle v_x \rangle = 2\frac{1}{T}\int_{t_1}^{0.5T} v_x \, dt \tag{2}$$
$$= 2\frac{V_s}{T}\left(\frac{T}{2} - t_1\right) \tag{3}$$
$$= 2\frac{V_s}{T}\left(\frac{T}{2} - \frac{L_c(2I_D)}{V_s}\right) \tag{4}$$
$$= V_s\left(1 - 2\frac{L_c(2I_D)}{V_sT}\right) \tag{5}$$

Clearly, V_D is a linear function of I_D and this load regulation curve can be plotted as in Fig. 3.

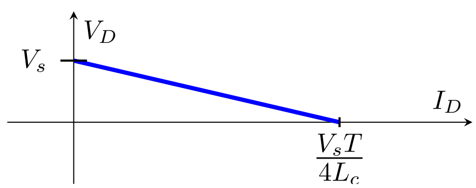

Figure 3: Load line for the full bridge with commutating inductance.

1.4 Current doubler rectifier

4.A

When the input voltage is positive, D1 is off and D2 is on. The current ID1 (we know this current is dc but we don't know what value it is yet) flows through the source, and both inductor currents are pulled through D2. When the input voltage is negative, the opposite is true. Thus v1 tracks the input while D1 is off, and v2 tracks the absolute value of the input while D2 is off. These waveforms are plotted in Fig. 4

4.B

To compute the power factor, we need the voltage and current at the input port. The voltage is fixed by the source, and we found before that the current is ID1 when the voltage is positive and ID2 when the voltage is negative. For ID1 = ID2, this results in a square wave with no dc component, totally

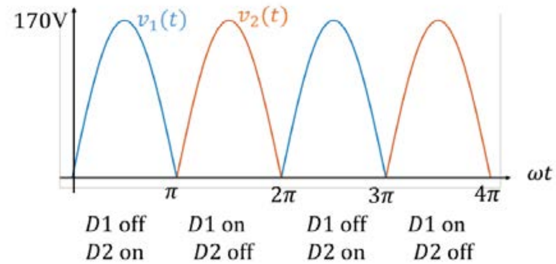

Figure 4: Diode voltages

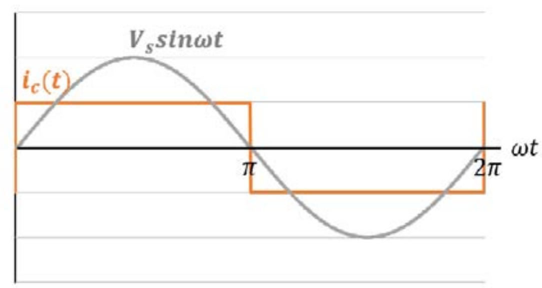

Figure 5: Current through the voltage source Lc = 0

in phase with the voltage. These waveforms are drawn in Fig. 5 for clarity.

Recall that power factor is the real power over the apparent power

$$PF = \frac{<p>}{VrmsIrms} = \frac{V_{1rms}I_{1rms}cos\theta}{V_{rms}I_{rms}} = \frac{I_{1rms}}{I_{rms}} \tag{6}$$

The current and voltage are in phase, so the displacement factor (cosine term) is equal to one. The total rms current is ID1. The fundamental amplitude of the current is $\frac{4}{\pi ID1}$, so the fundamental rms current is $\frac{4}{\pi}\frac{ID1}{\sqrt{2}}$ Therefore the power factor is $PF = \frac{4}{\pi\sqrt{2}} = \frac{2\sqrt{2}}{\pi} = 0.9$ Since the voltage and current are in phase, $k\theta = 1$ and therefore kd = PF = 0.9

4.C

By the definition of power factor, the maximum power that can be drawn from the ac source is $PFV_{rms}I_{maxrms} = (0.9)(\frac{170}{\sqrt{2}})(15) = 1620W.$ The maximum available power from the source is $V_{rms}I_{maxrms} = (\frac{170}{\sqrt{2}})(15) = 1800W.$. Any load with a power factor of 1, such as a resistor, could draw this power.

4.D

Things get worse if we include some commutating inductance. When the voltage changes signs, Lc does not allow the current through the source to change instantaneously. This forces both diodes to be on for some time. For example, just before the positive-to-negative voltage transition, D2 is on and carries ID1 + ID2 while Lc has ID1 flowing through it. The diode D1 is off and this state is valid because the diode has positive voltage across it. As soon as the voltage becomes negative, the condition we were using to know that D1 is off is no longer valid, so we expect this diode to be on. Current through the inductors cannot change instantly, thus D2 still carries ID1 + ID2. Thus, both diodes are on. The voltage drop across Lc becomes negative and its current decreases. While this is happening, D1 remains on and conducts current since the sum of the current through D1 and the current through Lc must always equal to ID1. Eventually, the current through the inductor will decrease to -ID2 at which point D2 will turn off (if the inductor current decreased any further, D2 would have to carry negative current, which is not possible). A similar commutation period occurs on the negative-to-positive voltage transition. During the transition, the voltage across the commutating inductance is $vc = V_s sin(\omega t)$. Therefore,

$$\begin{aligned} i_c &= \int_0^u \frac{V_s sin(\omega t) d(\omega t)}{\omega L_c} \\ &= \frac{V_s(1-cos(u)}{\omega L_c} + i_c(0) \\ &= \frac{V_s(1-cos(u)}{\omega L_c} - I_{D1} \end{aligned}$$

$$cosu = 1 - \frac{2\omega L_c I_{D1}}{V_s} \tag{7}$$

Now we can solve for the output voltage in terms of the output current. Since we are in periodic steady state, the average voltage across the inductors is zero. Thus

$$\begin{aligned} < V_D > =< V_1 > &=< V_2 > \\ &= \frac{1}{2\pi}\int_u^\pi V_s \sin(\omega t)\, d(\omega t) \\ &= \frac{Vs}{2\pi}(\cos u + 1) \\ &= \frac{Vs}{2\pi}\left(1 - \frac{2\omega L_c I_{D1}}{Vs} + 1\right) \\ &= \frac{Vs}{\pi} - \frac{\omega L_c I_{D1}}{\pi} \\ &= \frac{Vs}{\pi} - \frac{\omega L_c I_D}{2\pi} \end{aligned}$$

As expected, the output voltage droops as current is increased. To find maximum power, we could calculate power as a function of current, differentiate and set to zero. Or we can simply look at the Thevenin equivalent circuit and "impedance match" the load to the source resistance – this is the "famous maximum power transfer theorem." The Thevenin equivalent circuit having the same characteristics as our load-line is shown in Fig. 6 where

$$Rs = \frac{\frac{Vs}{\pi}}{\frac{2Vs}{\omega L_c}} = \frac{\omega L_c}{2\pi} \tag{8}$$

If we set RL equal to this value, then we will achieve maximum power transfer. The real power

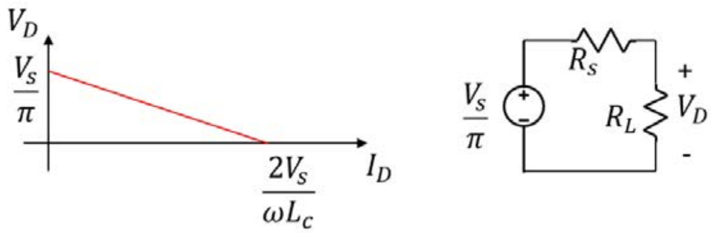

Figure 6: Load-line and equivalent circuit with non-zero commutating inductance.

at this point is simply

$$< p > max = \frac{V_L^2}{R_L}$$
$$= \frac{\left(\frac{Vs}{2\pi}\right)^2}{\frac{\omega L_c}{2\pi}} \quad = 4067W$$

Note that our Thevenin equivalent circuit is only valid in terms of average V-I characteristics. Even though our model has a current ID flowing through Rs no real dissipation occurs in that resistor ! Only the power delivered to R_L has real meaning and that is why we find $\langle$p$\rangle$max as the power consumed by R_L