6.622 HW8 Solutions*

8.1 Averaged Converter Model (Inverted Buck/Boost Circuit)

8.1.A Conversion Ratio

We can quickly find the converter's conversion ratio by using the fact that the PSS voltage across the inductor L is zero.

$$\langle v_L \rangle = V_1 D - V_c(1-D) = 0$$
$$\frac{V_c}{V_1} = \frac{D}{1-D}$$

8.1.B Average State-Space Equations

You can derive the average state-space equations using either state space averaging or direct circuit averaging. We'll show both methods here.

Method 1: State Space Averaging

$$\begin{aligned}
\frac{d\overline{i_L}}{dt} &= \frac{1}{L}\left[\overline{V_1 q(t)} - \overline{V_c(1-q(t))}\right] \\
&= \frac{1}{L}\left[\left(\overline{V_c} + \overline{V_1}\right) d - \overline{V_c}\right] \\
\frac{d\overline{V_c}}{dt} &= \frac{1}{C}\left[\overline{-\frac{V_c}{R}q(t)} + \overline{\left(i_L - \frac{V_c}{R}\right)(1-q(t))}\right] \\
&= \frac{1}{C}\left(\overline{i_L}(1-d) - \frac{\overline{V_c}}{R}\right) \\
&= \frac{1}{C}\left(\overline{i_L}d' - \frac{\overline{V_c}}{R}\right)
\end{aligned}$$

Method 2: Direct Circuit Averaging For our averaged circuit model, we replace the MOSFET with an averaged dependent current source and the diode with an averaged dependent voltage source, as shown in Fig. 1.

*Khandoker N. Rafa Islam 2023, adapted from Ranjram 2020

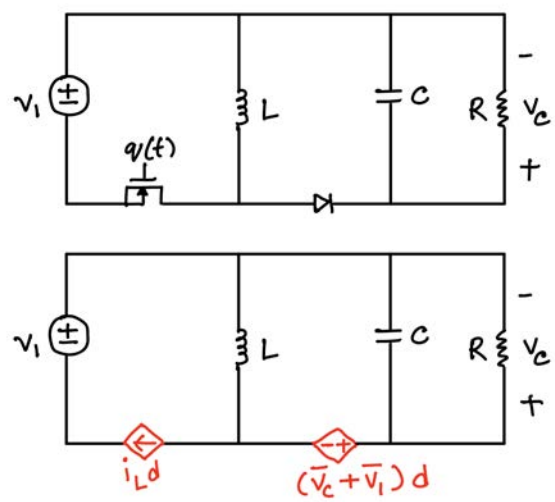

Figure 1: Averaged circuit model

This circuit model gives us the following average state-space equations.

$$\begin{aligned}
\frac{d\overline{i_L}}{dt} &= \frac{1}{L}\left[\left(\overline{V_c} + \overline{V_1}\right) d - \overline{V_c}\right] \\
&= \frac{1}{L}\left[\left(\overline{V_c} + \overline{V_1}\right) d - \overline{V_c}\right] \\
\frac{d\overline{V_c}}{dt} &= \frac{1}{C}\left[\left(\overline{i_L} - \overline{i_L}d\right) - \frac{\overline{V_c}}{R}\right] \\
&= \frac{1}{C}\left(\overline{i_L}d' - \frac{\overline{V_c}}{R}\right)
\end{aligned}$$

These equations are the same as the ones found using the state space averaging method.

8.1.C Averaged Circuit Model

One example of an averaged circuit model is shown in Fig. 1. Other averaged circuit models are also possible. For example, the MOSFET can be replaced with an averaged dependent voltage source and the diode can be replaced with an averaged dependent current source.

8.1.D Linearity

The averaged system is **not linear** because there are terms in the system that are products of the states and inputs.

8.2 Avg. model for Boost DCM

The diode current is triangular with height $V_{in}DT/L$. The base of this current is defined by the time it takes for this triangular current to decay to zero from this height. The slope of the current is

$$\begin{aligned} \frac{i_p}{\Delta t} &= -\frac{V_{in}-V_o}{L} \\ \Delta t &= \frac{V_{in}DT}{L}\frac{L}{V_o-V_{in}} \\ &= \frac{V_{in}DT}{V_o-V_{in}} \end{aligned}$$

The average current in the diode, which is equal to the average output current, is thus

$$\begin{aligned} \overline{i_d} &= \frac{1}{T}\left(\frac{i_p\Delta t}{2}\right) \\ &= \frac{1}{T}\left(\frac{1}{2}\frac{V_{in}DT}{L}\frac{V_{in}DT}{V_o-V_{in}}\right) \\ &= \frac{V_{in}^2D^2T}{2L\left(V_o-V_{in}\right)} \end{aligned}$$

8.3 Buck Converter Modeling

In class we learned how to derive the averaged and linearized model for controlling a boost converter, and we're now asked to do the same for a buck converter.

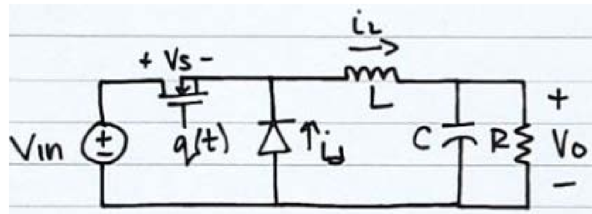

Figure 2: Circuit of a Buck Converter

8.3.A Averaged Circuit Modeling

The circuit of a typical buck converter is given in Fig. 2, and we find its averaged circuit.

For the switch, the voltage across the switch is

$$\begin{aligned} V_s(t) &= V_{in}(t)(1-q(t)) \\ \implies \overline{V_s(t)} &= \overline{V_{in}(t)(1-q(t))} \\ &\approx \overline{V_{in}(t)}\cdot d' \end{aligned}$$

where the approximation is valid if we assume both the input voltage $V_{in}(t)$ and the duty ratio $q(t)$ have small ripple and do not change drastically. $d' = 1-d$ is the complementary duty cycle.

Similarly for the diode current, with the same assumptions we can write

$$\begin{aligned} i_d(t) &= i_L(t)(1-q(t)) \\ \implies \overline{i_d(t)} &= \overline{i_L(t)(1-q(t))} \\ &\approx \overline{i_L(t)}\cdot d' \end{aligned}$$

Therefore, we can replace the two switches with their averaged model, and the averaged circuit is shown in Fig. 3.

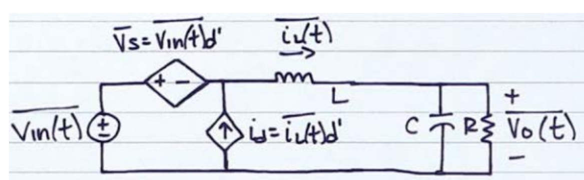

Figure 3: Averaged Buck Converter Circuit Model

We can then establish models for the averaged inductor current $\overline{i_L(t)}$ and averaged output voltage

$\overline{V_o(t)}$.

$$\begin{aligned}\frac{d\overline{i_L(t)}}{dt} &= \frac{\overline{V_{in}(t) - V_{in}(t)d' - V_o(t)}}{L} \\ &= \frac{\overline{V_{in}(t)}d}{L} - \frac{\overline{V_o(t)}}{L} \\ \frac{d\overline{V_o(t)}}{dt} &= \frac{\overline{i_L(t) - \frac{V_o(t)}{R}}}{C} \\ &= \frac{\overline{i_L(t)}}{C} - \frac{\overline{V_o(t)}}{RC}\end{aligned}$$

8.3.B State-Space Averaging

We identify the state variables being i_L and V_o, and we write

$$\begin{aligned}\frac{di_L}{dt} &= \frac{1}{L}[(V_{in}(t) - V_o(t))q(t) - V_o(t)(1 - q(t))] \\ &= \frac{V_{in}(t)q(t) - V_o(t)}{L} \\ \implies \frac{d\overline{i_L(t)}}{dt} &= \frac{\overline{V_{in}(t)}d}{L} - \frac{\overline{V_o(t)}}{L}\end{aligned}$$

and also

$$\begin{aligned}\frac{dV_o}{dt} &= \frac{1}{C}[(i_L(t) - V_o(t)/R)q(t) + \\ &\quad (i_L(t) - V_o(t)/R)(1 - q(t))] \\ &= \frac{i_L(t)}{C} - \frac{V_o(t)}{RC} \\ \implies \frac{d\overline{V_o(t)}}{dt} &= \frac{\overline{i_L(t)}}{C} - \frac{\overline{V_o(t)}}{RC}\end{aligned}$$

We find that the equations we get from direct circuit averaging and state-space averaging are exactly the same.

8.3.C Linearizion and Transfer Function

To linearize, we let

$$\begin{aligned}\overline{i_L(t)} &= I_L + \widetilde{i_L} \\ \overline{V_o(t)} &= V_o + \widetilde{v_o} \\ \overline{V_{in}(t)} &= V_{in} + \widetilde{v_{in}} \\ d &= D + \widetilde{d}\end{aligned}$$

and plug those into the averaged circuit model equations, note that for a buck converter in periodic steady-state,

$$\begin{aligned}V_o &= V_{in}D \\ I_L &= \frac{V_o}{R}\end{aligned}$$

Canceling out the P.S.S. terms and second order terms, we can simplify the linearized model as

$$\begin{aligned}\frac{d\widetilde{i_L}}{dt} &= \frac{V_{in}\widetilde{d} + \widetilde{v_{in}}D - \widetilde{v_o}}{L} \\ \frac{d\widetilde{v_o}}{dt} &= \frac{\widetilde{i_L}}{C} - \frac{\widetilde{v_o}}{RC}\end{aligned}$$

The transfer function from $\widetilde{d}$ to $\widetilde{v_o}$ can be found by setting the other perturbations $(\widetilde{V_{in}})$ to zero, and re-arranging the equations to get

$$\frac{\widetilde{v_o}}{\widetilde{d}} = \frac{V_{in}/LC}{s^2 + s/RC + 1/LC}$$

8.3.D Audio Susceptibility

The transfer function from $\widetilde{v_{in}}$ to $\widetilde{V_o}$ with duty ratio held constant can be similarly derived by setting $\widetilde{d} = 0$ in our linearized model, and hence we have

$$\frac{\widetilde{v_o}}{\widetilde{v_{in}}} = \frac{D/LC}{s^2 + s/RC + 1/LC}$$