6.642 Continuum Electromechanics
Fall 2008

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Useful Bessel Function Relationships

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MIT OpenCourseWare

Bessel Differential Equations

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(k^2x^2-p^2)y=0$$

or

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)+(k^2x^2-p^2)y=0$$

has solution

$$y=C_1J_p(kx)+C_2Y_p(kx).$$

$J_p(x)=$ Bessel function of first kind of order p.

$Y_p(x)=$ Bessel function of second kind of order p.

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(k^2x^2+p^2)y=0$$

or

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)-(k^2x^2+p^2)y=0$$

has solution

$$y=D_1J_p(jkx)+D_2Y_p(jkx)=E_1I_p(kx)+E_2K_p(kx).$$

$I_p(kx)=j^{-p}J_p(jkx)$ Modified Bessel function of first kind of order p,

$K_p(kx)=\frac{\pi}{2}j^{p+1}[J_p(jkx)+jY_p(jkx)]$ Modified Bessel function of second kind of order p,

$H_p^1(x)=J_p(x)+jY_p(x)$ Hankel function of first kind of order p,

$H_p^2(x)=J_p(x)-jY_p(x)$ Hankel function of second kind of order p.

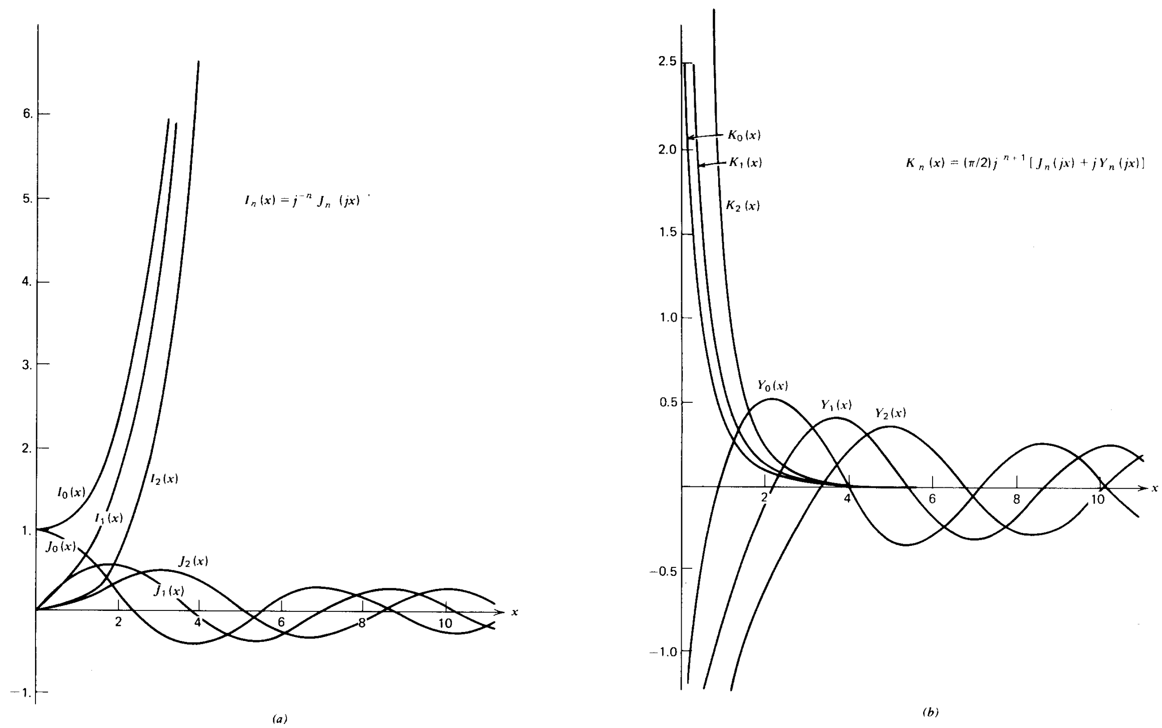

Figure 4-9 The Bessel functions (a) $J_n(x)$ and $I_n(x)$, and (b) $Y_n(x)$ and $K_n(x)$.

Derivatives of Bessel Functions

$$\frac{d}{dx}y_p(\alpha x) = \begin{cases} \alpha y_{p-1}(\alpha x) - \frac{p}{x}y_p(\alpha x) & [y = J, Y, I, H^1, H^2] \\ -\alpha y_{p-1}(\alpha x) - \frac{p}{x}y_p(\alpha x) & [y = K] \end{cases}$$

$$\frac{d}{dx}y_p(\alpha x) = \begin{cases} -\alpha y_{p+1}(\alpha x) + \frac{p}{x}y_p(\alpha x) & [y = J, Y, K, H^1, H^2] \\ \alpha y_{p+1}(\alpha x) + \frac{p}{x}y_p(\alpha x) & [y = I] \end{cases}$$

$$\left.\begin{array}{r} 2\frac{d}{dx}y_p(\alpha x) = \alpha[y_{p-1}(\alpha x) - y_{p+1}(\alpha x)] \\ y_{p-1}(\alpha x) + y_{p+1}(\alpha x) = \frac{2p}{\alpha x}y_p(\alpha x) \end{array}\right\} [y = J, Y, H^1, H^2]$$

$$2\frac{d}{dx}I_p(\alpha x) = \alpha[I_{p-1}(\alpha x) + I_{p+1}(\alpha x)]$$

$$2\frac{d}{dx}K_p(\alpha x) = -\alpha[K_{p-1}(\alpha x) + K_{p+1}(\alpha x)]$$

$$I_{p-1}(\alpha x) - I_{p+1}(\alpha x) = \frac{2p}{\alpha x}I_p(\alpha x)$$

$$K_{p-1}(\alpha x) - K_{p+1}(\alpha x) = -\frac{2p}{\alpha x}K_p(\alpha x)$$

$$\frac{d}{dx}y_0(\alpha x) = \begin{cases} -\alpha y_1(\alpha x) & [y = J, Y, K, H^1, H^2] \\ +\alpha y_1(\alpha x) & [y = I] \end{cases}$$