6.642 Continuum Electromechanics
Fall 2008

Lecture 11: Smoothly Inhomogeneous Systems
Continuum Electromechanics (Melcher) – Section 8.18

I. Governing Equations

$$\rho\left[\frac{\partial \bar{v}}{\partial t}+\left(\bar{v}\cdot\nabla\right)\bar{v}\right]=-\nabla p+q\bar{E}-\frac{1}{2}E^2\nabla\varepsilon-\rho g\,\bar{i}_x$$

$$\frac{\partial \rho}{\partial t}+\left(\bar{v}\cdot\nabla\right)\rho=0\Rightarrow\nabla\cdot\bar{v}=0,\ \frac{\partial \rho}{\partial t}+\left(\bar{v}\cdot\nabla\right)\rho=0\ \text{(Incompressible)}$$

$$\nabla\cdot\bar{J}+\frac{\partial q}{\partial t}=0,\ \bar{J}=q\bar{v}\Rightarrow\frac{\partial q}{\partial t}+\nabla\cdot\left(q\bar{v}\right)=\frac{\partial q}{\partial t}+\left(\bar{v}\cdot\nabla\right)q=0$$

(q=charge density, $\nabla\cdot\bar{v}=0$)
Perfectly insulating fluid

$$\frac{\partial \varepsilon}{\partial t}+\left(\bar{v}\cdot\nabla\right)\varepsilon=0$$

(Permittivity tied to fluid)

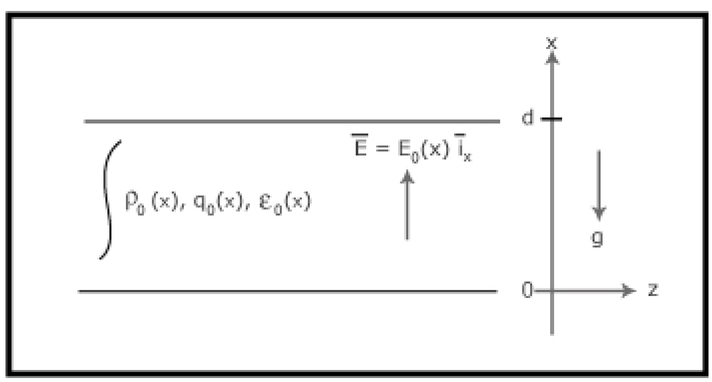
$$\frac{D\varepsilon}{Dt}=\frac{\partial \varepsilon}{\partial t}+\left(\bar{v}\cdot\nabla\right)\varepsilon=0$$

$$\nabla\cdot\left(\varepsilon\bar{E}\right)=q$$

II. Linearization

$$\rho_0\frac{\partial \bar{v}'}{\partial t}=-\rho' g\,\bar{i}_x-\nabla p'+q'\bar{E}_0+q_0\bar{e}-\frac{1}{2}E_0^2\nabla\varepsilon'-\bar{E}_0\cdot\bar{e}\,\nabla\varepsilon_0$$

$$\nabla\cdot\bar{v}'=0$$



$$\frac{\partial \rho'}{\partial t}+\left(\bar{v}'\cdot\nabla\right)\rho_0=0$$

$$\frac{\partial \varepsilon'}{\partial t}+\left(\bar{v}'\cdot\nabla\right)\varepsilon_0=0$$

$$\frac{\partial q'}{\partial t}+\left(\bar{v}'\cdot\nabla\right)q_0=0$$

$$\nabla\cdot\left(\varepsilon'\bar{E}_0\right)+\nabla\cdot\left(\varepsilon_0\bar{e}\right)=q'$$

$$\bar{E}=E_0\,\bar{i}_x+\bar{e},\quad \bar{e}=-\nabla\Phi',\quad \Phi'=\mathrm{Re}\left[\hat{\Phi}(x)\,e^{j(\omega t-kz)}\right]$$

$$\bar{v} = 0 + \bar{v}'$$

$$\rho = \rho_0(x) + \rho'$$

$$q = q_0(x) + q'$$

$$\varepsilon = \varepsilon_0(x) + \varepsilon'$$

$$j\omega\hat{\rho} + \hat{v}_x \frac{d\rho_0}{dx} = 0 \Rightarrow \hat{\rho} = -\frac{\hat{v}_x}{j\omega}\frac{d\rho_0}{dx}$$

$$\hat{q} = -\frac{\hat{v}_x}{j\omega}\frac{dq_0}{dx}$$

$$\hat{\varepsilon} = -\frac{\hat{v}_x}{j\omega}\frac{d\varepsilon_0}{dx}$$

Momentum Equations

$$\rho_0 j\omega\hat{v}_x = -\hat{\rho} g - \frac{d\hat{p}}{dx} + \hat{q}E_0(x) - q_0\frac{d\hat{\Phi}}{dx} - \frac{1}{2}E_0^2(x)\frac{d\hat{\varepsilon}}{dx} + E_0(x)\frac{d\hat{\Phi}}{dx}\frac{d\varepsilon_0}{dx}$$

$$\rho_0 j\omega\hat{v}_z = jk\hat{p} + q_0(x)\,jk\hat{\Phi} + \frac{1}{2}E_0^2(x)\,jk\hat{\varepsilon}$$

$$\frac{d\hat{v}_x}{dx} - jk\hat{v}_z = 0 \Rightarrow \hat{v}_z = \frac{1}{jk}\frac{d\hat{v}_x}{dx}$$

$$\frac{d}{dx}\left(\hat{\varepsilon}E_0(x)\right) + \frac{d}{dx}\left(\varepsilon_0\left(-\frac{d\hat{\Phi}}{dx}\right)\right) + jk\left(-jk\hat{\Phi}\right)\varepsilon_0 = \hat{q}$$

$$\hat{p} = \frac{1}{jk}\left[\rho_0 j\omega\hat{v}_z - q_0 jk\hat{\Phi} - \frac{1}{2}jkE_0^2\hat{\varepsilon}\right]$$

$$= \frac{1}{jk}\left[\frac{\rho_0 \not{j}\omega}{\not{j}k}\frac{d\hat{v}_x}{dx} - q_0 jk\hat{\Phi} - \frac{1}{2}jkE_0^2\hat{\varepsilon}\right]$$

$$\rho_0 j\omega\hat{v}_x = -\hat{\rho} g - \frac{1}{jk}\left[\frac{\omega}{k}\frac{d}{dx}\left(\rho_0\frac{d\hat{v}_x}{dx}\right) - jk\frac{d}{dx}\left(q_0\hat{\Phi}\right)\frac{-jk}{2}\frac{d}{dx}\left(E_0^2\hat{\varepsilon}\right)\right] + \hat{q}E_0 - q_0\frac{d\hat{\Phi}}{dx}$$
$$- \frac{1}{2}E_0^2\frac{d\hat{\varepsilon}}{dx} + E_0\frac{d\hat{\Phi}}{dx}\frac{d\varepsilon_0}{dx}$$

$$j\omega\left[\frac{d}{dx}\left(\rho_0\frac{d\hat{v}_x}{dx}\right)-\rho_0 k^2\hat{v}_x\right]=-k^2 g\frac{d\rho_0}{dx}\frac{\hat{v}_x}{j\omega}-k^2\hat{\Phi}\frac{dq_0}{dx}+k^2E_0\frac{dq_0}{dx}\frac{\hat{v}_x}{j\omega}+k^2E_0\frac{dE_0}{dx}\frac{d\varepsilon_0}{dx}\frac{\hat{v}_x}{j\omega}$$

$$-k^2E_0\frac{d\varepsilon_0}{dx}\frac{d\hat{\Phi}}{dx}$$

Gauss' Law

$$-\frac{d}{dx}\left[\varepsilon_0\frac{d\hat{\Phi}}{dx}\right]+k^2\varepsilon_0\hat{\Phi}+\hat{\varepsilon}\frac{dE_0}{dx}+E_0\frac{d\hat{\varepsilon}}{dx}=-\frac{dq_0}{dx}\frac{\hat{v}_x}{j\omega}$$

$$\frac{d}{dx}\left[\varepsilon_0\frac{d\hat{\Phi}}{dx}\right]-k^2\varepsilon_0\hat{\Phi}+\frac{dE_0}{dx}\frac{d\varepsilon_0}{dx}\frac{\hat{v}_x}{j\omega}+\frac{E_0}{j\omega}\frac{d}{dx}\left(\hat{v}_x\frac{d\varepsilon_0}{dx}\right)=\frac{dq_0}{dx}\frac{\hat{v}_x}{j\omega}$$

III. Exchange of Stability

A. Momentum Equation

Multiply by $\hat{v}_x^*$ and integrate from x=0 to x=d

Define:

$$I_1=-\int_0^d\hat{v}_x^*\left[\frac{d}{dx}\left(\rho_0\frac{d\hat{v}_x}{dx}\right)-\rho_0k^2\hat{v}_x\right]dx$$

$$=\underbrace{-\hat{v}_x^*\rho_0\frac{d\hat{v}_x}{dx}\Big|_0^d}_{=0}+\int_0^d\left[\rho_0\frac{d\hat{v}_x}{dx}\frac{d\hat{v}_x^*}{dx}+\rho_0k^2\hat{v}_x\hat{v}_x^*\right]dx$$

$$=\int_0^d\rho_0\left[\left|\frac{d\hat{v}_x}{dx}\right|^2+k^2\left|\hat{v}_x\right|^2\right]dx \qquad \text{(positive definite and real)}$$

$$I_2=k^2\int_0^d\left[g\frac{d\rho_0}{dx}-E_0\frac{dq_0}{dx}-E_0\frac{dE_0}{dx}\frac{d\varepsilon_0}{dx}\right]\left|\hat{v}_x\right|^2dx \qquad \text{(real)}$$

$$I_3=k^2\int_0^d\hat{v}_x^*\left[\frac{dq_0}{dx}\hat{\Phi}+E_0\frac{d\varepsilon_0}{dx}\frac{d\hat{\Phi}}{dx}\right]dx$$

$$j\omega I_1=\frac{I_2}{j\omega}+I_3$$

B. Gauss' Law

$$I_4 = -k^2 \int_0^d \hat{\Phi}^* \left[\frac{d}{dx}\left(\varepsilon_0 \frac{d\hat{\Phi}}{dx} \right) - k^2 \varepsilon_0 \hat{\Phi} \right] dx$$

$$= \cancelto{0}{-k^2 \hat{\Phi}^* \varepsilon_0 \frac{d\hat{\Phi}}{dx}\Big|_0^d} + k^2 \int_0^d \left[\varepsilon_0 \frac{d\hat{\Phi}}{dx} \frac{d\hat{\Phi}^*}{dx} + k^2 \varepsilon_0 \hat{\Phi}\hat{\Phi}^* \right] dx$$

$$= k^2 \int_0^d \varepsilon_0 \left[\left| \frac{d\hat{\Phi}}{dx} \right|^2 + k^2 \left| \hat{\Phi} \right|^2 \right] dx \text{ (positive definite and real)}$$

$$I_5 = k^2 \int_0^d \hat{\Phi}^* \left[\frac{d}{dx}\left(E_0 \frac{d\varepsilon_0}{dx} \hat{v}_x \right) - \frac{dq_0}{dx} \hat{v}_x \right] dx$$

$$= \cancelto{0}{k^2 \hat{\Phi}^* E_0 \frac{d\varepsilon_0}{dx} \hat{v}_x \Big|_0^d} - k^2 \int_0^d \left[E_0 \frac{d\varepsilon_0}{dx} \hat{v}_x \frac{d\hat{\Phi}^*}{dx} + \hat{\Phi}^* \frac{dq_0}{dx} \hat{v}_x \right] dx$$

$$= -I_3^* \Rightarrow -I_4 + \frac{I_5}{j\omega} = 0$$

C. Stability

$$I_5 = -I_3^* = j\omega I_4$$

$$-\omega^2 I_1 = I_2 + j\omega I_3 = I_2 + j\omega\left(j\omega^*\right) I_4 = -\omega\omega^* I_4 + I_2$$

$$-\omega^2 I_1 - I_2 + \omega\omega^* I_4 = 0$$

I_1 and I_4 positive definite real, I_2 (real)

$$\omega = \alpha + j\beta, \ \omega^* = \alpha - j\beta$$

$$-\left(\alpha^2 - \beta^2 + 2j\alpha\beta\right) I_1 - I_2 + \left(\alpha^2 + \beta^2\right) I_4 = 0$$

$$\alpha\beta I_1 = 0$$

$$-\left(\alpha^2 - \beta^2\right) I_1 - I_2 + \left(\alpha^2 + \beta^2\right) I_4 = 0$$

Either $\alpha = 0$ or $\beta = 0$: $e^{j\omega t} = e^{j(\alpha + j\beta)t} = e^{j\alpha t} e^{-\beta t} \Rightarrow \beta > 0$ Stable

$$\alpha = 0, \ \beta > 0$$

$$\beta^2\left(I_1 + I_4\right) = I_2 \Rightarrow \beta^2 = \frac{I_2}{I_1 + I_4}$$

$$\beta = \pm\sqrt{\frac{I_2}{I_1 + I_4}}$$

For β real $\Rightarrow I_2 > 0$ (Unstable when β negative)

Sufficient Condition for Stability $\left(I_2 < 0\right)$

$$g\frac{d\rho_0}{dx} - E_0\frac{dq_0}{dx} - E_0\frac{dE_0}{dx}\frac{d\varepsilon_0}{dx} < 0$$

IV. Weak Gradient Case Study (Boussinesq approximation)

$$\rho_0\left(x\right) = \rho_T\left(1 + \alpha x\right) \quad , \; \alpha d \ll 1$$

$$q_0\left(x\right) = q_T\left(1 + \beta x\right) \quad , \; \beta d \ll 1, \; q_T \text{ small}$$

$$\varepsilon_0\left(x\right) = \varepsilon_T \text{ (constant)}$$

$$E_0\left(x\right) = E_T \text{ (constant)}$$

$$\rho_T j\omega\left[\frac{d^2\hat{v}_x}{dx^2} - k^2\hat{v}_x\right] = -\frac{\hat{v}_x k^2}{j\omega}\left[g\rho_T\alpha - E_T\beta q_T\right] - k^2\hat{\Phi}q_T\beta$$

$$\frac{d^2\hat{\Phi}}{dx^2} - k^2\hat{\Phi} = -\frac{\hat{v}_x}{j\omega}\frac{q_T\beta}{\varepsilon_T}$$

$$\hat{v}_x = \hat{V}e^{rx} \;, \; \hat{\Phi} = \hat{F}e^{rx}$$

$$\omega_m^2 = \frac{E_T\beta q_T - \alpha\rho_T g}{\rho_T}; \; \omega_p^2 = \frac{\left(\beta q_T\right)^2 d^2}{\rho_T\varepsilon_T}$$

$$\frac{d^2\hat{v}_x}{dx^2} - k^2\hat{v}_x + \frac{k^2\hat{v}_x}{\omega^2}\omega_m^2 + \frac{\beta q_T k^2}{\rho_T j\omega}\hat{\Phi} = 0$$

$$\frac{-q_T\beta}{j\omega\varepsilon_T}\hat{v}_x + \frac{d^2\hat{\Phi}}{dx^2} - k^2\hat{\Phi} = 0$$

$$\hat{V}\left[r^2 - k^2 + k^2\frac{\omega_m^2}{\omega^2}\right] + \frac{\beta q_T k^2}{\rho_T j\omega}\hat{F} = 0$$

$$\frac{-q_T\beta}{j\omega\varepsilon_T}\hat{V} + \left(r^2 - k^2\right)\hat{F} = 0$$

$$\left[r^2 - k^2\right]\left[r^2 - k^2 + k^2\frac{\omega_m^2}{\omega^2}\right] - \frac{\left(q_T\beta\right)^2 k^2}{\omega^2\rho_T\varepsilon_T} = 0$$

$$\left[r^2 - k^2\right]^2 + \left[r^2 - k^2\right]\frac{k^2\omega_m^2}{\omega^2} - \frac{\omega_p^2 k^2}{\omega^2 d^2} = 0$$

$$r^2 - k^2 = -\frac{k^2\omega_m^2}{2\omega^2} \pm \frac{1}{2}\sqrt{\left(\frac{k^2\omega_m^2}{\omega^2}\right)^2 + \frac{4\omega_p^2 k^2}{\omega^2 d^2}}$$

$$r_{1,2,3,4} = \pm\left[k^2 - \frac{k^2\omega_m^2}{2\omega^2} \pm \frac{1}{2}\sqrt{\left(\frac{k^2\omega_m^2}{\omega^2}\right)^2 + \frac{4\omega_p^2 k^2}{\omega^2 d^2}}\right]^{1/2}$$

$$r_2 = -r_1$$

$$r_4 = -r_3$$

$$\hat{\Phi} = \Phi_1 e^{r_1 x} + \Phi_2 e^{r_2 x} + \Phi_3 e^{r_3 x} + \Phi_4 e^{r_4 x}$$

$$\hat{v}_x = \frac{j\omega}{q_T\beta}\left[\left(r_1^2 - k^2\right)\Phi_1 e^{r_1 x} + \left(r_2^2 - k^2\right)\Phi_2 e^{r_2 x} + \left(r_3^2 - k^2\right)\Phi_3 e^{r_3 x} + \Phi_4\left(r_4^2 - k^2\right)e^{r_4 x}\right]$$

Boundary Conditions:

$$\hat{\Phi}\left(x = 0\right) = 0 \qquad \hat{v}_x\left(x = 0\right) = 0$$
$$\hat{\Phi}\left(x = d\right) = 0 \qquad \hat{v}_x\left(x = d\right) = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{r_1 d} & e^{-r_1 d} & e^{r_3 d} & e^{-r_3 d} \\ \left(r_1^2 - k^2\right) & \left(r_1^2 - k^2\right) & \left(r_3^2 - k^2\right) & \left(r_3^2 - k^2\right) \\ \left(r_1^2 - k^2\right)e^{r_1 d} & \left(r_1^2 - k^2\right)e^{-r_1 d} & \left(r_3^2 - k^2\right)e^{r_3 d} & \left(r_3^2 - k^2\right)e^{-r_3 d} \end{bmatrix}\begin{bmatrix}\Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4\end{bmatrix} = 0$$

$$4\left(r_3^2 - r_1^2\right)\sinh r_3 d \;\; \sinh r_1 d = 0$$

$$r_3^2 = -\left(\frac{n\pi}{d}\right)^2 \quad \text{or} \quad r_1^2 = -\left(\frac{n\pi}{d}\right)^2$$

$$-\left[\left(\frac{n\pi}{d}\right)^2 + k^2\right] = \frac{-k^2\omega_m^2}{2\omega^2} \pm \frac{1}{2}\sqrt{\left(\frac{k^2\omega_m^2}{\omega^2}\right)^2 + \frac{4\omega_p^2 k^2}{\omega^2 d^2}}$$

$$4\left\{\frac{k^2\omega_m^2}{2\omega^2} - \left[\left(\frac{n\pi}{d}\right)^2 + k^2\right]\right\}^2 = \frac{1}{\not{4}}\left[\left(\frac{k^2\omega_m^2}{\omega^2}\right)^2 + \frac{4\omega_p^2 k^2}{\omega^2 d^2}\right]$$

$$4\left[\left(\frac{k^2\omega_m^2}{2\omega^2}\right)^2 + \left[\left(\frac{n\pi}{d}\right)^2 + k^2\right]^2 - \frac{k^2\omega_m^2}{\omega^2}\left[\left(\frac{n\pi}{d}\right)^2 + k^2\right]\right] = \left(\frac{k^2\omega_m^2}{\omega^2}\right)^2 + \frac{4\omega_p^2 k^2}{\omega^2 d^2}$$

$$\left[\left(\frac{n\pi}{d}\right)^2 + k^2\right]^2 = \frac{1}{\omega^2}\left[k^2\omega_m^2\left[\left(\frac{n\pi}{d}\right)^2 + k^2\right] + \frac{\omega_p^2 k^2}{d^2}\right]$$

$$\omega^2 = \frac{k^2\left[\omega_m^2\left[\left[\frac{n\pi}{d}\right]^2 + k^2\right] + \frac{\omega_p^2}{d^2}\right]}{\left[\left(\frac{n\pi}{d}\right)^2 + k^2\right]^2}$$

$$\omega_p^2 > 0,\ \omega_m^2 > 0 \quad \text{if } \beta q_T E_T > \alpha\rho_T g \qquad \left(\text{Stable}\left(\omega^2 > 0\right)\right)$$

$$\omega_m^2 < 0 \quad \text{if } \beta q_T E_T < \alpha\rho_T g \qquad (\text{Destabilizing})$$