Generalizations: Adding an extra parameter

In the discussion of implementing the Fourier Transform and the Inverse Fourier Transform, $\Psi(x)$ and $A(q)$, were always described as vectors, indexed by **x** and **q** respectively. In general, these each could have been described by a matrix, again, with a row index of of **x** and **q**, but with a column index of some independent parameter, like time.

For example, $\mathbf{\Psi}$ could have been a matrix. Each column could be for a different time instance:

$$\mathbf{\Psi} = \begin{pmatrix} \Psi(x_1,t_1) & \Psi(x_1,t_2) & \cdots & \Psi(x_1,t_l) \\ \Psi(x_2,t_1) & \Psi(x_2,t_2) & \cdots & \Psi(x_2,t_l) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi(x_n,t_1) & \Psi(x_n,t_2) & \cdots & \Psi(x_n,t_l) \end{pmatrix}$$

then, applying the Fourier Transform procedures exactly as described previously, we would get a matrix $\mathbf{A}(q,t)$ instead of the vector $\vec{A}(q)$:

$$\mathbf{A}(q,t) = \begin{pmatrix} A(q_1,t_1) & A(q_1,t_2) & \cdots & A(q_1,t_l) \\ A(q_2,t_1) & A(q_2,t_2) & \cdots & A(q_2,t_l) \\ \vdots & \vdots & \ddots & \vdots \\ A(q_m,t_1) & A(q_m,t_2) & \cdots & A(q_m,t_l) \end{pmatrix}$$

The most probable scenario is that we are given the initial wavepacket $\Psi(x,t=0)$ and wish to find the wave packet at time **t>0**. In this case we perform the Fourier Transform on a single column, and are returned a single column amplitude function. Then, we want to find $\mathbf{\Psi}$ for many time instances. Using matrices instead of vectors, we can compute all the time instances at once. First, we setup a matrix $\mathbf{A}$:

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} A(q_1) \\ A(q_2) \\ \vdots \\ A(q_m) \end{pmatrix} * \begin{pmatrix} 1 & 1 & \cdots & 1_m \end{pmatrix} \\ &= \begin{pmatrix} A(q_1) & A(q_1) & \cdots & A(q_1) \\ A(q_2) & A(q_2) & \cdots & A(q_2) \\ \vdots & \vdots & \ddots & \vdots \\ A(q_m) & A(q_m) & \cdots & A(q_m) \end{pmatrix} \end{aligned}$$

This is a matrix, with identical column. Each column is the expsansion coefficients we computed from the Fourier transform of $\mathbf{\Psi}$. We will put one column in the matrix for each future time instance we wish to compute $\mathbf{\Psi}$ at.

If we then similarly redefine $\vec{\zeta}$ to account for the time parameter as follows:

$$\vec{\zeta} = \begin{pmatrix} e^{-iE_1 t_1/\hbar} & e^{-iE_1 t_2/\hbar} & \cdots & e^{-iE_1 t_l/\hbar} \\ e^{-iE_2 t_1/\hbar} & e^{-iE_2 t_2/\hbar} & \cdots & e^{-iE_2 t_l/\hbar} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-iE_m t_1/\hbar} & e^{-iE_m t_2/\hbar} & \cdots & e^{-iE_m t_l/\hbar} \end{pmatrix}$$

Now, if we perform array multiplication (element by element) on ζ and $\mathbf{A}$ we get:

$$\mathbf{A}(q,t) = \begin{pmatrix} A(q_1)e^{-it_1E_1/\hbar} & A(q_1)e^{-it_2E_1/\hbar} & A(q_1)e^{-it_3E_1/\hbar} & \cdots & A(q_1)e^{-it_lE_1/\hbar} \\ A(q_2)e^{-it_1 1E_2/\hbar} & A(q_2)e^{-it_2E_2/\hbar} & A(q_2)e^{-it_3E_2/\hbar} & \cdots & A(q_2)e^{-it_lE_2/\hbar} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(q_m)e^{-it_1E_m/\hbar} & A(q_m)e^{-it_2E_m/\hbar} & A(q_m)e^{-it_3E_m/\hbar} & \cdots & A(q_m)e^{-it_lE_m/\hbar} \end{pmatrix}$$

and, if we multiply $\vec{\phi}_{\vec{q}} * \mathbf{A}(q,t)$, just like we did in the previous sections:

$$= \begin{pmatrix} (A(q_1,t_1)e^{iq_1x_1} + \cdots + A(q_m,t_1)e^{iq_mx_1}) & \cdots & (A(q_1,t_l)e^{iq_1x_1} + \cdots + A(q_m,t_l)e^{iq_m} \\ (A(q_1,t_1)e^{iq_1x_2} + \cdots + A(q_m,t_1)e^{iq_mx_2}) & \cdots & (A(q_m,t_l)e^{iq_1x_2} + \cdots + A(q_m,t_l)e^{iq_m} \\ \vdots & \vdots & \vdots \\ (A(q_1,t_1)e^{iq_1x_n} + \cdots + A(q_m,t_1)e^{iq_mx_n}) & \cdots & (A(q_1,t_l)e^{iq_1x_n} + \cdots + A(q_m,t_l)e^{iq_m} \end{pmatrix}$$

which is just

$$\Psi(x,t) = \begin{pmatrix} \Psi(x_1,t_1) & \Psi(x_1,t_2) & \cdots & \Psi(x_1,t_l) \\ \Psi(x_2,t_1) & \Psi(x_2,t_2) & \cdots & \Psi(x_2,t_l) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi(x_n,t_1) & \Psi(x_n,t_2) & \cdots & \Psi(x_n,t_l) \end{pmatrix}$$