

$$\psi = \widehat{\psi} \, e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

Then a good guess of the differential equation that gives the dispersion relation is

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$$

This guess is justified by experimental confirmation; this is *not* a derivation.

Massachusetts Institute of Technology– 6.763 2005 Lecture 9

Schrödinger's Equation (with forces)

We present a plausibility argument, not a derivation, relating the classical formulation to the quantum formulation.

The energy for a particle in a force is, classically,

$$\mathcal{E} = \frac{1}{2}m \left(\mathbf{v} \cdot \mathbf{v}\right) + V(\mathbf{r})$$

Energy is conserved since the potential is independent of time.

$$0 = \frac{d\mathcal{E}}{dt} = m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{d}{dt}V(\mathbf{r})$$

= $m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{\partial}{\partial t}V(\mathbf{r}) + (\mathbf{v} \cdot \nabla)V(\mathbf{r})$
= $\mathbf{v} \cdot \left(m\frac{d\mathbf{v}}{dt} + \nabla V\right) \longrightarrow m\frac{d\mathbf{v}}{dt} = -\nabla V$
Massachusetts Institute of Technology
6.763 2005 Lecture 9

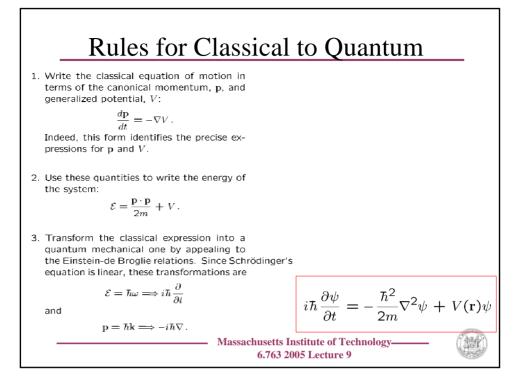
$$Canonical Momentum & Schrödinger's Equation
$$\frac{d\mathbf{p}}{dt} = -\nabla V$$

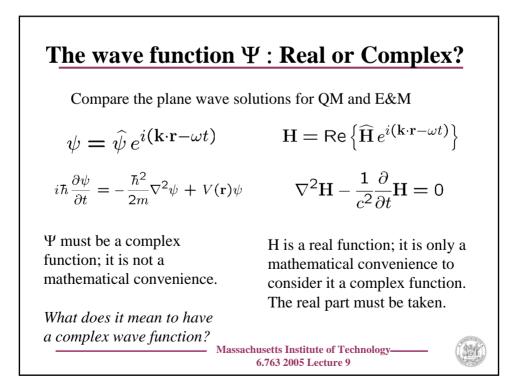
$$\frac{d}{dt} (\text{canonical momentum}) = -\nabla (\text{generalized potential})$$

$$\mathbf{p} = m\mathbf{v} \qquad \mathbf{V}$$
Here, the canonical momentum equals the kinematic
momentum; and the generalized potential, the scalar potential.

$$\mathcal{E} = \frac{1}{2m} (\mathbf{p} \cdot \mathbf{p}) + V(\mathbf{r})$$

$$\hbar\omega = \frac{\hbar^2}{2m} (\mathbf{k} \cdot \mathbf{k}) + V(\mathbf{r}) \implies i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{r}) \psi$$
Schrödinger's Equation
Massachusetts Institute of Technology
6763 2005 Lecture 9$$





The physical meaning of the wave function Ψ

The absolute phase of a plane wave should *not* influence the overall physics of a system.

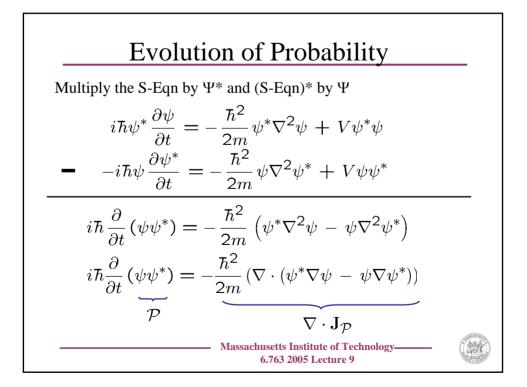
So Max Born hypothesized in ~1927 that the square of the magnitude of the wave function Ψ was equal to the *probability* of a quantum mechanical particle to be at the location **r** at time t.

$$\wp(\mathbf{r},t) \equiv |\psi(\mathbf{r},t)|^2 = \psi^*(\mathbf{r},t)\psi(\mathbf{r},t)$$

With the normalization condition (particle must be somewhere)

$$\int d\mathbf{r} \, \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) = 1$$

Massachusetts Institute of Technology— 6.763 2005 Lecture 9



Probability Current

Therefore we find that the probability

$$\wp(\mathbf{r},t) \equiv |\psi(\mathbf{r},t)|^2 = \psi^*(\mathbf{r},t)\psi(\mathbf{r},t)$$

and the probability current

$$\mathbf{J}_{\wp} \equiv \frac{\hbar}{2im} (\psi^* \nabla \psi - \psi \nabla \psi^*) = \operatorname{Re} \left\{ \psi^* \frac{\hbar}{im} \nabla \psi \right\}$$

satisfy a continuity relation

$$\frac{\partial \wp}{\partial t} = -\nabla \cdot \mathbf{J}_{\wp}$$

 Massachusetts Institute of Technology– 6.763 2005 Lecture 9

Schrödinger's Equation with E&M Fields

For a charged particle, we want the classical equations such that

 $\frac{d}{dt}$ (canonical momentum) = $-\nabla$ (generalized potential)

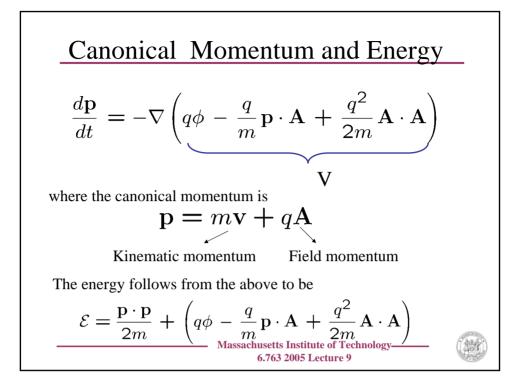
Start with the Lorentz Force Law

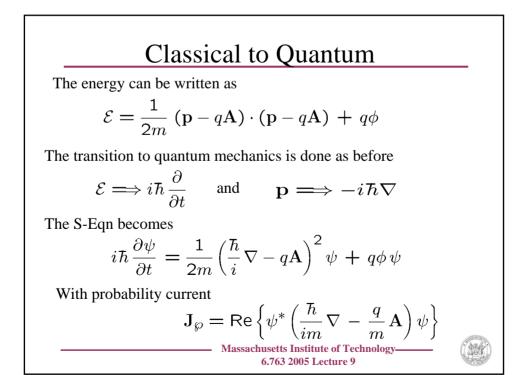
$$m \frac{d\mathbf{v}}{dt} = q \left(\mathbf{E} + (\mathbf{v} \times \mathbf{B}) \right)$$

and use the vector and scalar potentials

$$\mathbf{B} = \nabla \times \mathbf{A}$$
 and $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$ to find

Massachusetts Institute of Technology— 6.763 2005 Lecture 9





Macroscopic Quantum ModelA specific describes the whole ensemble of superelectrons such that $\psi^*(\mathbf{r},t)\psi(\mathbf{r},t) = n^*(\mathbf{r},t) \rightarrow density$ and $\int d\mathbf{r} \ \psi^*(\mathbf{r},t)\psi(\mathbf{r},t) = \mathbb{N}^* \longrightarrow \text{ Total number}$ Colspan="2">Optimized current density given byJac $= q^* \ \operatorname{Re} \left\{ \psi^*\left(\frac{\hbar}{im^*} \nabla - \frac{q^*}{m^*} \mathbf{A}\right) \psi \right\}$

DESCRIPTION J. This macroscopic quantum wavefunction follows $i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \frac{1}{2m^{\star}} \left(\frac{\hbar}{i} \nabla - q^{\star} \mathbf{A}(\mathbf{r}, t)\right)^{2} \Psi(\mathbf{r}, t) + q^{\star} \phi(\mathbf{r}, t) \Psi(\mathbf{r}, t)$ Writing $\Psi(\mathbf{r}, t) = \sqrt{n^{\star}(\mathbf{r}, t)} e^{i\theta(\mathbf{r}, t)}$, we find $J_{\mathsf{S}} = q^{\star} n^{\star}(\mathbf{r}, t) \left(\frac{\hbar}{m^{\star}} \nabla \theta(\mathbf{r}, t) - \frac{q^{\star}}{m^{\star}} \mathbf{A}(\mathbf{r}, t)\right)$ **DESCRIPTION Massachusetts Institute of Technology**

The Supercurrent Equation

$$\mathbf{J}_{\mathsf{S}} = q^{\star} n^{\star}(\mathbf{r}, t) \left(\frac{\hbar}{m^{\star}} \nabla \theta(\mathbf{r}, t) - \frac{q^{\star}}{m^{\star}} \mathbf{A}(\mathbf{r}, t) \right)$$

 \mathbf{J}_{s} is a unique physical quantity, but \mathbf{A} and θ are not.

If a new vector and scalar potential are found in another gauge such that

