Area, Perimeter & Euler Number in Continuous Gray-level Images

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It is well known that area, perimeter, and Euler numbers are the three "locally computable" properties of discrete binary images. That is, they can be computed simply by adding up the results of local Boolean computations over the image region of interest — such local computations can proceed in parallel [Gray, 1971].

The creation of a binary image is plagued by the difficulty of classifying a pixel as "background" (0) or "object" (1) — something that cannot be done reliably, particularly at the edge of the object where pixels straddle the boundary (sometimes called "partial pixels"). Then it may be reasonable instead to assign a number ranging from 0 to 1 indicating how much of the pixel is likely to be part of the object (or what the probability is that the pixel is part of the object). However, discrete Boolean operators are then no longer appropriate, since pixel values are no longer two-valued.

Further, it may be of interest to consider what happens in the limit as the pixel size is made smaller and smaller. In this case we arrive at a continuous function of image position. Discrete sums of locally computed values must then be replaced with integrals.

A question then arises: if we let $0 \le B(x,y) \le 1$ be a continuous function of image position (x,y), is there an equivalent to the "local computation" of area, perimeter and Euler number? Here we assume that $B(x,y)$ has rapid, but smooth transitions between the two levels — i.e. the first and second partial derivatives exist.

The answer to the question above is then provided by the formulae:

$$A = \iint B(x,y)\, dx\, dy$$

$$P = \iint \sqrt{B_x^2 + B_y^2}\, dx\, dy$$

$$E = -\frac{1}{2\pi} \iint \frac{B_{xx}B_y^2 - 2B_{xy}B_xB_y + B_{yy}B_x^2}{B_x^2 + B_y^2}\, dx\, dy$$

where (B_x, B_y) is the brightness gradient and B_{xx}, B_{xy}, and B_{yy} are the second partial derivatives of brightness.

The integrand in the expression for P is the first derivative of image brightness in the direction of the brightness gradient. The integrand in the expression for E is the second derivative of image brightness in the direction perpendicular to the brightness gradient.

In the case of discrete images, the integrals above can be approximated using finite sums, and the derivatives can be approximated using finite differences.

Interestingly, in the original discrete binary image domain [Horn, 1986]:

(i) the local computation of area can be done using information from individual pixels directly;

(ii) the local computation of perimeter requires information from pairs of adjacent picture cells; and

(iii) the local computation of Euler number requires information from three picture cells meeting at a vertex in the picture cell grid.

Correspondingly, in the continuous gray-level image domain:

(i) area can be computed using the image values directly;

(ii) perimeter requires first partial derivatives (gradient); while

(iii) the Euler number requires second order partial derivatives.

References

Gray, S.B. (1971) “Local Properties of Binary Images in Two Dimensions,” IEEE Trans. on Computers, Vol. 20, No. 5, pp. 551–561, May.

Horn, B.K.P. (1986) *Robot Vision*, MIT Press & McGraw-Hill