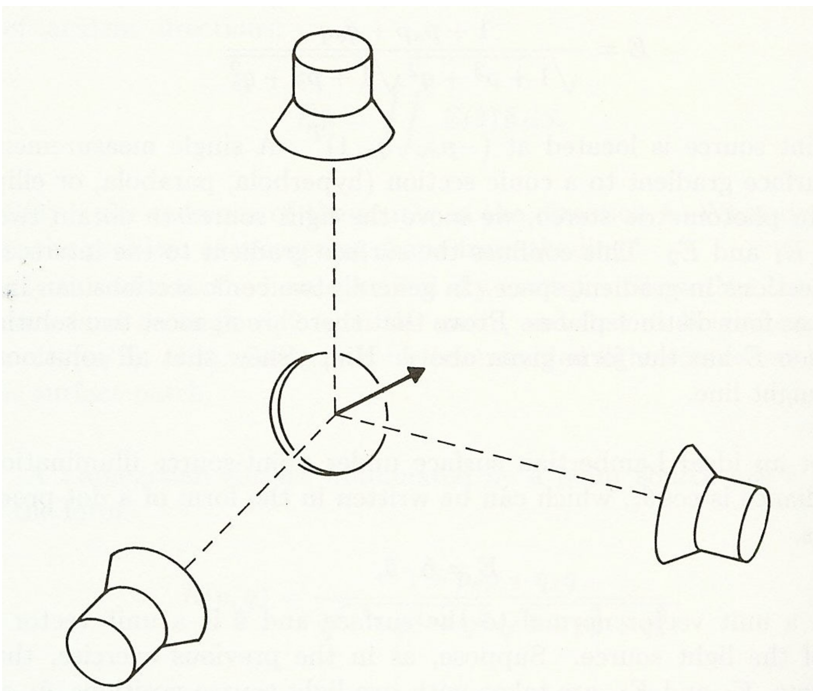
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF COMPUTER SCIENCE AND ELECTRICAL ENGINEERING

6.801/6.866 MACHINE VISION



Problem 1: This problem is about improving the accuracy of the photometric stereo method for recovering surface shape. We might want to take more than two brightness measurements in order to improve accuracy when using the photometric stereo method. Imagine that n light sources are used in turn to obtain n images. Suppose that the surface under consideration is Lambertian and that the direction to the i-th source is given by the unit vector $\hat{\mathbf{s}}_i$. Assume that the surface can have an albedo ρ (where $0 < \rho < 1$) (i.e. it is not neccessarily an *ideal* Lambertian surface). At each point in the image, we wish to find the unit surface normal $\hat{\mathbf{n}}$ that minimizes the sum of squares of errors

$$\sum_{i=1}^{n} (\rho\, \hat{\mathbf{n}} \cdot \hat{\mathbf{s}}_i - E_i)^2 ,$$

where E_i is the i-th measurement of brightness at that point, and ρ the albedo.

(a) Show that the vector that minimizes the sum of squares of errors is

$$\rho\,\hat{\mathbf{n}} = \left[\sum_{i=1}^{n} \hat{\mathbf{s}}_i \hat{\mathbf{s}}_i^T\right]^{-1} \sum_{i=1}^{n} E_i \hat{\mathbf{s}}_i,$$

where $\mathbf{a}\mathbf{b}^T$ is the *dyadic product* of the vectors $\mathbf{a}$ and $\mathbf{b}$, i.e.

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \begin{pmatrix} b_x & b_y & b_z \end{pmatrix} = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix},$$

and $[\;]^{-1}$ indicates the inverse of the matrix.

(b) Show that the matrix is singular when there is only one measurement ($n = 1$). How about two measurements ($n = 2$)?

(c) What is the smallest number n of measurements needed to guarantee that the indicated matrix inverse exists? **Hint:** This part of the problem is nontrivial.

(d) How would you expect the minimum number of measurements needed change if it were known that the albedo was one ($\rho = 1$)?

Problem 2: This problem is about a family of surface reflectance models used to describe light reflection off the surfaces of rocky planets, asteroids, and satellites. A phenomenological reflectance model for such surfaces is an ideal 'Minnaert' surface for which

$$f(\theta_i, \phi_i; \theta_e, \phi_e) = (c+1)/(2\pi)\;(\cos\theta_i \cos\theta_e)^{c-1}$$

for $0 \le c \le 1$. Here, when illuminated by a distant point source, brightness is given by

$$L = (c+1)/(2\pi)\; E_0 \cos^{c-1}\theta_e \cos^c \theta_i$$

(a) Does the surface reflectance so defined obey Helmholtz reciprocity?

(b) For what value of c is the surface brightness independent of the *viewing* direction?

(c) For what value of c is the surface brightness independent of the *illumination* direction?

(d) Is Lambert's formula a special case of Minnaert's formula?

(f) Is a Hapke-type surface a special case of a Minnaert surface?

(f) Is a surface with radiance proportional to $\sec\theta_e$ a special case also?

Problem 3: This problem is about ambiguity in recovering the shape of an object. Suppose that the reflectance map is linear in p and q, so that

$$R(p, q) = ap + bq + c.$$

(for example, it could be a Hapke-type surface). We have an image, including the silhouette of a simple convex object of shape $z = f(x, y)$. Show that the surface

$$\overline{z} = f(x, y) + g(bx - ay),$$

for an *arbitrary* differentiable function $g(s)$, will give rise to the *same* image. Does the surface $\overline{z}$ have the same silhouette? Assume that the derivative of g is bounded.

Problem 4: This problem is about starting the solution of a shape-from-shading problem at a singular point, by fitting a smooth local shape near the singular point. Consider the image of a polynomial surface with a stationary point at the origin:

$$z(x, y) = \alpha(x^2 + y^2) + 2\beta xy$$

Assume that the rotationally symmetric reflectance map is simply

$$R(p, q) = (1 + p^2 + q^2)$$

(a) What is the image $E(x, y)$ of this surface? What is the brightness gradient (E_x, E_y)?

(b) What is the relationship between the second order partial derivates E_{xx} and E_{yy}? What is the relationship between the second order partial derivatives and the coefficients α and β of the polynomial?

(c) Given measurements of E_{xx}, E_{xy}, and E_{yy} at the origin, how many solutions (values of α and β) are there for the surface shape?

(d) Suppose that an image of such a polynomial surface has a non-zero mixed second order partial derivative E_{xy}. Is it possible to rotate the coordinate system so that in the mixed partial derivative is zero in the rotated system?

Hint: Express the second partial derivatives in the rotated coordinate system in terms of the ones in the original coordinate system — then try and find a rotation that makes the mixed derivative drop out.

Problem 5: This problem is about "line" detection — as opposed to "edge" detection. The brightness gradient $\nabla E = (E_x, E_y)$ is useful in recovering image motion. It is also useful for "edge" detection where we look for extrema of the brightness gradient along the direction of the brightness gradient itself.

The first directional derivative of brightness along a line that makes an angle θ with the x axis is

$$\frac{dE}{ds} = E_x \cos\theta + E_y \sin\theta$$

The *second* derivatives of brightness are of use in recovering local surface curvature, as well as for "line" detection.

(a) Show that the maximum directional *first* derivative is in the direction given by the unit gradient vector

$$(E_x, E_y) \Big/ \sqrt{E_x^2 + E_y^2}$$

and that the slope in that direction is $\sqrt{E_x^2 + E_y^2}$.

(b) Show that the *second* directional derivative along a line that makes an angle θ with the x axis is:

$$\frac{d^2 E}{ds^2} = E_{xx} \cos^2\theta + 2E_{xy} \sin\theta\cos\theta + E_{yy} \sin^2\theta.$$

(c) Show that the second directional derivative has extrema for

$$\tan 2\theta = 2\, E_{xy} \big/ (E_{xx} - E_{yy})$$

(d) Show that the extreme values are

$$E''_{\text{min,max}} = \frac{1}{2}(E_{xx} + E_{yy}) \pm \frac{1}{2}\sqrt{(E_{xx} - E_{yy})^2 + 4\,E_{xy}^2}.$$

e) Show that the four directions of extrema in second directional derivative are given by:

$$\pm\left(\sqrt{\frac{D \pm (E_{xx} - E_{yy})}{2D}}, \operatorname{sign}(E_{xy})\sqrt{\frac{D \mp (E_{xx} - E_{yy})}{2D}}\right).$$

wher

$$D = \sqrt{(E_{xx} - E_{yy})^2 + 4E_{xy}^2}.$$

(f) When will there *not* be extrema in the directional second derivative?

(g) What is the geometric relationship between the gradient direction $\nabla E = (E_x, E_y)$ and the direction of the minimum second directional derivative on 'top of a ridge' of the brightness surface? What is the geometric relationship between the gradient direction and the direction of the maximum second directional derivative at the 'bottom of a valley' of the brightness surface?

Hints: Use identities for trigonometric functions of doubled angles. Express $\sin 2\theta$ and $\cos 2\theta$ in terms of $\tan 2\theta$. Substitute back into the expression for the directional second derivative. Express $\cos\theta$ and $\sin\theta$ in terms of $\cos 2\theta$ and $\sin 2\theta$. Note that the second directional derivative can be positive or negative.