Massachusetts Institute of Technology

Department of Computer Science and Electrical Engineering

6.801/6.866 Machine Vision

Problem 1: Here we consider some properties of a two-dimensional version of the absolute orientation problem (that is, we are dealing with alignment of images rather than volumes). Let (x_i, y_i) and (x'_i, y'_i) be coordinates of corresponding points in two images. For convenience, assume that the coordinates are measured relative to the centroids of the points in the two images (i.e. the sum of the coordinates in each image is zero). We are to minimize the sum of squares of errors in the transform:

$$\frac{1}{\sqrt{s}}x'_i = \sqrt{s}(+\cos\theta x_i + \sin\theta y_i)$$
$$\frac{1}{\sqrt{s}}y'_i = \sqrt{s}(-\sin\theta x_i + \cos\theta y_i)$$

by suitable choice of the 'symmetrical' scale factor s and the rotation θ. (Note that translation has already been taken care of by using coordinates referenced to the centroids).

(a) Show that when formulated this way, the total error to be minimized consists of a term that depends only on s, and not on θ, and a term that depends only on θ, and not on s — thus separting the problem of finding the rotation from that of finding the scaling.

(b) Show that the best fit value for s can be computed without knowing the rotation θ — *and* without knowing the correspondences between points in the two coordinate systems!

(c) Show that the best fit value for θ can be found without knowing s.

(d) The resulting equation for θ appears to have more than one solution. Do all of the solutions minimize the sum of squares of errors in image position?

(e) What was the advantage of 'splitting' the contribution of scale s into two factors, $\sqrt{s}$ and $1/\sqrt{s}$ (rather than just using say $x'_i = s\ldots$ and $y'_i = s\ldots$)?

Problem 2: This problem is about relating different ways of representing rotation in 3-D. Rodrigues' formula provides a method for rotating a vector $\mathbf{r}$ about an axis $\hat{\boldsymbol{\omega}}$ through an angle θ (see figure on next page):

$$\mathbf{r}' = (\cos\theta)\mathbf{r} + (1-\cos\theta)(\mathbf{r}\cdot\hat{\boldsymbol{\omega}})\hat{\boldsymbol{\omega}} + (\sin\theta)(\hat{\boldsymbol{\omega}}\times\mathbf{r})$$

(a) Can a rotation through an angle $-\theta$ about a different axis producce the same result? What is that axis?

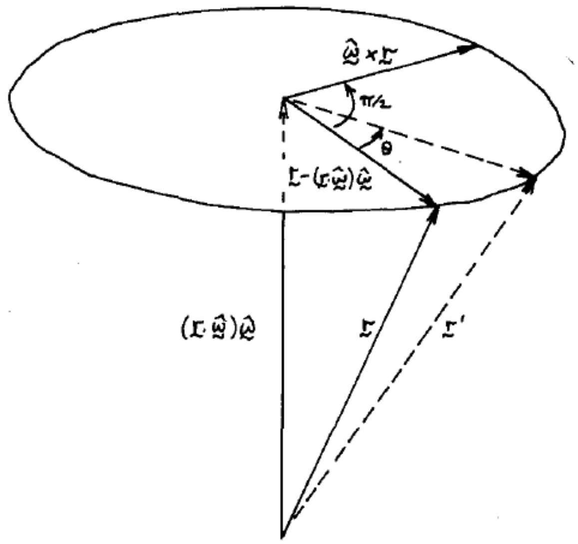

Fig. 6. The rotation of a vector **r** to a vector **r**′ can be understood in terms of the quantities appearing in this diagram. Rodrigues's formula follows.

(b) Use Rodrigues' rotation formula to show that the rotation matrix R for a rotation about an axis specified by the unit vector $\hat{\boldsymbol{\omega}}$ through an angle θ can be written

$$R = \cos\theta\ I + (1 - \cos\theta)\ \hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}}^T + \sin\theta\ \hat{\boldsymbol{\omega}}_\times$$

where $\hat{\boldsymbol{\omega}}_\times$ is a 3×3 skew symmetric matrix such that $\hat{\boldsymbol{\omega}}_\times \mathbf{v} = \hat{\boldsymbol{\omega}} \times \mathbf{v}$ for all $\mathbf{v}$.

(c) Express Trace(R) in terms of $\cos\theta$ and $\sin\theta$, where Trace(R) is the sum of the diagonal elements of the matrix R.

(d) How could you recover θ and $\hat{\boldsymbol{\omega}}$ from R (hint: consider using Trace(R), and the elements of $R + R^T$ and $R - R^T$ to recover $\cos\theta$, $\sin\theta$, and $\hat{\boldsymbol{\omega}}_\times$).

(e) Show that $\hat{\boldsymbol{\omega}}_\times^2 = \hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}}^T - I$ (where $\hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}}^T$ is the dyadic product of $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{\omega}}$).

(f) Show that the matrix R defined in (b) is in fact orthonormal ($R^T R = I$).

(g) Based on (b), conclude that we can write the rotation matrix in terms of the corresponding unit quaternion $\mathring{q} = (q, \mathbf{q})^T$ as

$$R = (q^2 - \mathbf{q} \cdot \mathbf{q})I + 2\mathbf{q}\mathbf{q}^T + 2q\mathbf{q}_\times$$

where $\mathbf{q}_\times$ is the skew symmetric matrix such that $\mathbf{q}_\times \mathbf{v} = \mathbf{q} \times \mathbf{v}$ for all v.

(h) Express Trace(R) in terms of q and $\mathbf{q}$ (may or may not need to remember that $q^2 + \mathbf{q} \cdot \mathbf{q} = 1$).

(i) How would you recover q and $\mathbf{q}$ from R (hint: consider using Trace(R), and the elements of $R + R^T$ and $R - R^T$ in analogy with part (d) above).

Problem 3: Problems in photogrammetry require recovery of Euclidean motions, that is, translation and rotation. In this problem we prove some quaternion identities that are helpful in using quaternions to represent rotations in three dimensions. Remember that the product of two quaternions $\mathring{a} = (a, \mathbf{a})$ and $\mathring{b} = (a, \mathbf{b})$ can be written

$$(a, \mathbf{a})(b, \mathbf{b}) = (ab - \mathbf{a} \cdot \mathbf{b}, a\mathbf{b} + b\mathbf{a} + \mathbf{a} \times \mathbf{b})$$

The conjugate of a quaternion $\mathring{a} = (a, \mathbf{a})$ is $\mathring{a}^* = (a, -\mathbf{a})$. The dot-product of two quaternions $\mathring{a}$ and $\mathring{b}$ can be written $(a, \mathbf{a}) \cdot (b, \mathbf{b}) = ab + \mathbf{a} \cdot \mathbf{b}$.

(a) Show that

$$\mathring{a}\mathring{a}^* = \mathring{a}^*\mathring{a} = (\mathring{a} \cdot \mathring{a})\mathring{e}$$

where $\mathring{e}$ is a quaternion with unit scalar part and zero vector part.

(b) Show that

$$(\mathring{a}\mathring{b}) \cdot (\mathring{a}\mathring{b}) = (\mathring{a} \cdot \mathring{a})(\mathring{b} \cdot \mathring{b})$$

(c) Show that

$$(\mathring{a}\mathring{b}) \cdot (\mathring{a}\mathring{c}) = (\mathring{a} \cdot \mathring{a})(\mathring{b} \cdot \mathring{c})$$

(d) Show that

$$(\mathring{a}\mathring{q}) \cdot \mathring{b} = \mathring{a} \cdot (\mathring{b}\mathring{q}^*) \quad \text{and} \quad \mathring{a} \cdot (\mathring{q}\mathring{b}) = (\mathring{q}^*\mathring{a}) \cdot \mathring{b}$$

(e) Is the following always true?

$$(\mathring{q}\mathring{a}\mathring{q}^*) \cdot (\mathring{q}\mathring{b}\mathring{q}^*) = \mathring{a} \cdot \mathring{b}$$

Is a constraint on $\mathring{q}$ needed?

(f) How about?

$$(\mathring{q}\mathring{a}\mathring{q}^*) \cdot \big((\mathring{q}\mathring{b}\mathring{q}^*)(\mathring{q}\mathring{c}\mathring{q}^*)\big) = \mathring{a} \cdot (\mathring{b}\mathring{c})$$

If necessary, assume that $\mathring{a}$, $\mathring{b}$, and $\mathring{c}$, are quaternions representing vectors, that is, with zero scalar part — and, if necessary, that $\mathring{q}$ is a unit quaternion.

Problem 4: Here we explore an alternate way of determining rigid body motion from two sets of 3-D measurements. Suppose that we have measured the coordinates of n known features on an object before movement, and found them to be $\{\boldsymbol{\ell}_i\}$ for $i = 1, 2 \ldots n$, and then measured them again after movement, and found them to be $\{\mathbf{r}_i\}$ for $i = 1, 2 \ldots n$. One way of estimating the rigid body motion that took place is to find the transformation that: (i) moves the average $\bar{\boldsymbol{\ell}}$ of $\{\boldsymbol{\ell}_i\}$ to the average $\bar{\mathbf{r}}$ of $\{\mathbf{r}_i\}$, and (ii) rotates the "axes of inertia" of the two sets of measurements into exact alignment.

Let $\boldsymbol{\ell}'_i = \boldsymbol{\ell}_i - \bar{\boldsymbol{\ell}}$ and $\mathbf{r}'_i = \mathbf{r}_i - \bar{\mathbf{r}}$ be coordinates relative to the centroids.

(a) Now consider the inertia of the "point cloud" $\{\mathbf{r}'_i\}$ about an axis defined by the unit vector $\hat{\boldsymbol{\omega}}$ (through the average $\bar{\mathbf{r}}$). Show that the square of the perpendicular distance of the point $\mathbf{r}'_i$ from the axis of rotation is

$$\mathbf{r}'_i \cdot \mathbf{r}'_i - (\hat{\boldsymbol{\omega}} \cdot \mathbf{r}'_i)^2$$

(b) Show that overall the inertia about the axis $\hat{\boldsymbol{\omega}}$ can be expressed in the form

$$I(\hat{\boldsymbol{\omega}}) = \sum_{i=1}^{n} \mathbf{r}_i'^T \mathbf{r}_i' - \hat{\boldsymbol{\omega}}^T \left(\sum_{i=1}^{n} \mathbf{r}_i' \mathbf{r}_i'^T \right) \hat{\boldsymbol{\omega}}$$

Is the 3×3 matrix that appears in the above expression symmetric?

(c) The first term in the above sum is a constant, equal to the sum of squares of distances of the points from the origin, and does not depend on the choice of the axis $\hat{\boldsymbol{\omega}}$. Show that the stationary values (maxima, minima and saddle points) of the inertia are given by the eigenvalues of the 3×3 "inertia matrix" in the above expression.

Correspondingly, the axis directions that give rise to these stationary values are the eigenvectors of this matrix. The three eigenvectors will be orthogonal to each other if the eigenvalues are distinct. Suppose the eigenvectors are $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ corresponding to eigenvalues λ_1, λ_2, and λ_3 where $\lambda_1 > \lambda_2 > \lambda_3 > 0$.

(d) As an example, suppose that $n = 6$ and that the measured points are

$$(a, 0, 0)^T, (-a, 0, 0)^T, (0, b, 0)^T, (0, -b, 0)^T, (0, 0, c)^T, (0, 0, -c)^T$$

where $a > b > c > 0$. Determine the inertia matrix, and find its eigenvalues and eigenvectors.

(e) Show that in general any vector $\mathbf{r}$ can be expressed as a linear combination $\mathbf{r}' = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ or, in other words,

$$\mathbf{r}' = \left(\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \right) \mathbf{a} = M_r \mathbf{a}$$

where $\mathbf{a} = (a_1, a_2, a_3)^T$, and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the three eigenvectors of part (c).

(f) The same analysis can be applied to the other measurements $\{\boldsymbol{\ell}_i\}$ to obtain the eigenvectors $\mathbf{e}_1'$, $\mathbf{e}_2'$, and $\mathbf{e}_3'$ of a 3×3 "inertia matrix." So we can express any vector $\boldsymbol{\ell}$ as a linear combination $\boldsymbol{\ell}' = b_1\mathbf{e}_1' + b_2\mathbf{e}_2' + b_3\mathbf{e}_3'$ or

$$\boldsymbol{\ell}' = \left(\mathbf{e}_1' \mathbf{e}_2' \mathbf{e}_3' \right) \mathbf{b} = M_l \mathbf{b}$$

where $\mathbf{b} = (b_1, b_2, b_3)^T$. Express the orthonormal rotation matrix R in the relation $\mathbf{r}' = R(\boldsymbol{\ell}')$ in terms of the two matrices M_r and M_l. Is the result orthonormal?

(g) Do correspondences between points need to be known when using this method? Can this method accomodate an arbitrary number of measured points? When will this method of estimating absolute orientation fail?

(h) What is the smallest number of points that needs to be measured? (Keep in mind that the centroid is first subtracted out …)

Problem 5: Here we consider a machine vision method for determining the attitude of a spacecraft. Suppose we wish to use a camera in a spacecraft to determine the attitude from an image of a star field. We assume, first of all, that star images have been matched with stars in a catalog (see e.g. *Lisp* by Winston and Horn). Since the stars are essentially infinitely far away, the images we obtain are not affected by the position of the craft, only its attitude.

(a) Let us first solve an equivalent problem in two dimensions. We are given catalog directions (l_{xi}, l_{y_i}) matched with observed directions (r_{xi}, r_{y_i}) and have to find the angle of rotation θ that will carry the later into best alignment with the former. Explain why maximizing

$$\sum_{i=0}^{n-1} (l_{xi}, l_{y_i}) \cdot \mathrm{Rot}\big((r_{xi}, r_{y_i})\big)$$

is a reasonable strategy. Show that this is equivalent to maximizing

$$c \sum_{i=0}^{n-1} \big(l_{xi} r_{xi} + l_{y_i} r_{y_i}\big) + s \sum_{i=0}^{n-1} \big(l_{xi} r_{y_i} - l_{y_i} r_{xi}\big)$$

subject to $c^2 + s^2 = 1$. Find the solution for c and s.

(b) Now let us solve the three-dimensional version of the problem. In analogy with the two-dimensional case, we wish to maximize

$$\sum_{i=0}^{n-1} \boldsymbol{\ell}_i \cdot \mathrm{Rot}(\mathbf{r}_i) = \sum_{i=0}^{n-1} \mathring{\ell}_i \cdot \mathring{q}\, \mathring{r}_i\, \mathring{q}^*$$

subject to $\mathring{q} \cdot \mathring{q} = 1$, where $\mathring{\ell}_i = (0, \boldsymbol{\ell}_i)$ and $\mathring{r}_i = (0, \mathbf{r}_i)$. Show that this is the same as maximizing

$$\sum_{i=0}^{n-1} \mathring{q}\mathring{r}_i \cdot \mathring{\ell}_i \mathring{q}$$

subject to $\mathring{q} \cdot \mathring{q} = 1$.

(c) Use the fact that

$$\mathring{\ell}\, \mathring{q} = \mathfrak{L}\, \mathring{q} \quad \text{and} \quad \mathring{q}\, \mathring{r} = \overline{\mathfrak{R}}\, \mathring{q}$$

for some orthogonal 4×4 matrices $\mathfrak{L}$ and $\overline{\mathfrak{R}}$ to reduce this to an eigenvector/eigenvalue problem.