

Problem Set 1 Solutions

by Rick Cory and John Roberts

1 Definition of Underactuated

- a) Without loss of generality, assume there is no damping in the system and the mass and inertia are equal to one. The manipulator equations of the submarine are then:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \mathbf{B} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \quad (1)$$

where $[x \ y \ \theta]$ are the generalized coordinates (θ is the angle of the submarine from horizontal), and T_i represents the thrust generated by thruster i . Since the thrust axes are 30 degrees away from each other, the total force vector acting on the submarine is given by:

$$\mathbf{F} = \begin{bmatrix} T_1 \cos(\theta + \frac{\pi}{6}) + T_2 \cos \theta + T_3 \cos(\theta - \frac{\pi}{6}) \\ T_1 \sin(\theta + \frac{\pi}{6}) + T_2 \sin \theta + T_3 \sin(\theta - \frac{\pi}{6}) \\ 0 \end{bmatrix} \quad (2)$$

Since the forces act at the point of intersection of the three thrust axes, the torque about the center of mass (com) is given by:

$$\tau_{\text{com}} = \begin{bmatrix} 2 \cos \theta \\ 2 \sin \theta \\ 0 \end{bmatrix} \times \mathbf{F} = \begin{bmatrix} 0 \\ 0 \\ T_1 - T_3 \end{bmatrix} \quad (3)$$

Thus we can re-write equation 1 as:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta + \frac{\pi}{6}) & \cos \theta & \cos(\theta - \frac{\pi}{6}) \\ \sin(\theta + \frac{\pi}{6}) & \sin \theta & \sin(\theta - \frac{\pi}{6}) \\ 1 & 0 & -1 \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \quad (4)$$

where we used the identity $\sin(u - v) = \sin u \cos v - \sin v \cos u$. When θ is a multiple of π or an odd multiple of $\frac{\pi}{2}$, the matrix \mathbf{B} will not be full ranked. Hence in these states, the system is underactuated.

- b) Assuming unit mass/inertia and no damping, the manipulator equations for the robot are:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \mathbf{B} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad (5)$$

Since B is a 3×2 matrix, then $\text{rank}(B) \leq 2 < \dim(\mathbf{x})$, therefore the system is underactuated.

- c) All nonholonomic systems are underactuated because their constraints on velocity can be differentiated to obtain constraints on acceleration. For example, in part (b) above differentiating the velocity constraint gives us:

$$\frac{d}{dt}(\dot{y} \cos \theta - \dot{x} \sin \theta) = 0 \quad (6)$$

$$\Rightarrow \ddot{y} \cos \theta - \dot{y} \sin \theta - \dot{x} \sin \theta - \dot{x} \cos \theta = \dot{\theta} \sqrt{\dot{y}^2 + \dot{x}^2} \quad (7)$$

Hence the orientation of the robot imposes a constraint on the instantaneous accelerations.

- d) The telescope system is always fully actuated in ϕ and θ as they can be directly controlled by the input torques, as is clear in the equations of motion. In x and y , however, there is a kinematic singularity at $\phi = 0$, which precludes the system from accelerating in any direction save that in which the telescope is currently pointing (i.e., $\dot{y}/\dot{x} = \tan\theta$). Therefore the system is underactuated for some configuration ($\phi = 0$) when trying to control x and y .

2 The Simple Pendulum

- a) For the case of $b = u = 0$ the basin of attraction has a single point; namely the fixed point at $[0 \ 0]^T$. The fixed point (closest to the origin) for the case $b = 0.5, u = 0$ is $[0 \ 0]^T$. The fixed point for the case $b = 0.5, u = \frac{g}{2l}$ is $[\frac{\pi}{6} \ 0]$. The basins of attraction for the latter two cases are shown in Figure 1.

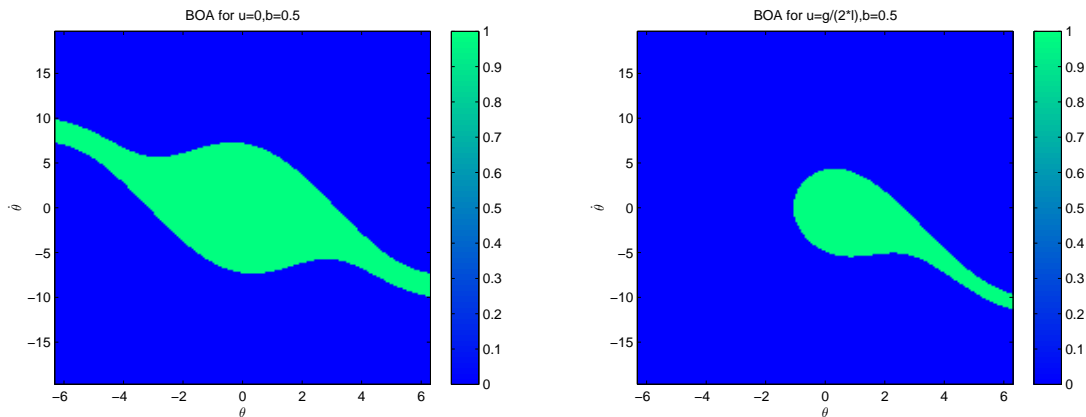


Figure 1: The basins of attraction (green) for the case of $b = 0.5, u = 0$ (left) and $b = 0.5, u = \frac{g}{2l}$ (right)

- b) The trajectories for the system with $u = 0$ and with $u = b\dot{\theta}$ are shown in Figure 2. Note that the feedback linearized system ($u = b\dot{\theta}$) creates a closed circuit (as would an undamped pendulum), with integration errors accounting for any small wobble seen in the trajectory. The system with no torque spirals in towards the fixed point, as would be expected given the damping.

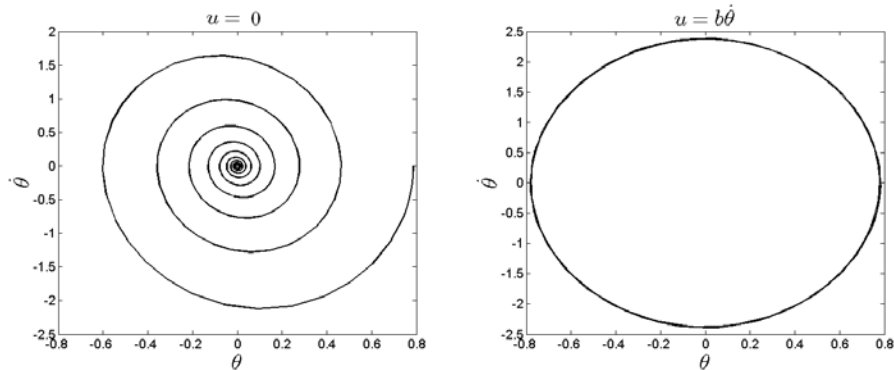


Figure 2: Trajectories for the system without torque (left) and the system feedback linearized to be without damping (right).

- c) Doubling gravity will result in the phase plot of the now effectively undamped system “stretching” vertically, as higher speeds will be reached, but it will remain qualitatively the same. Torque as large as mgI is required to double gravity, while torque as large as $2mgI$ is required to invert it, as double gravity just requires the motor to create an “extra” gravity, while inverting requires first canceling gravity, then adding another negative gravity in.

3 Optimal control of the double integrator

- a) The trajectory followed by the min-time policy is much sharper and reaches higher speeds than those achieved by the LQR. The LQR follows a much smoother path to the goal, and accelerates much less due to the cost put on actuation, while the min-time policy is constantly using as much of its actuator as it can.

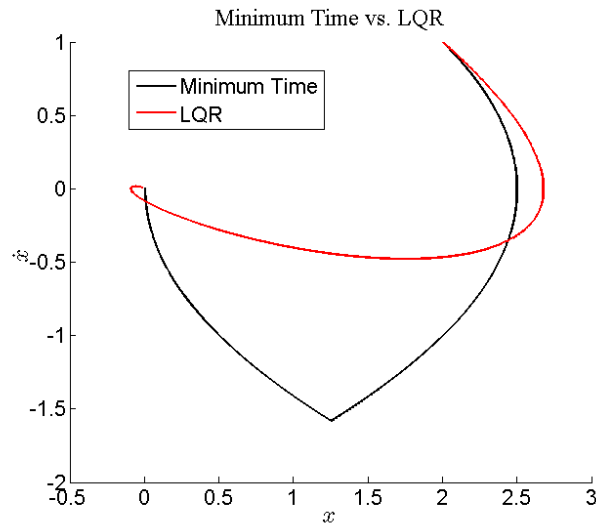


Figure 3: Trajectories for the system following the minimum time and the LQR policies.

- b) See part a.
- c) This solution assumes that the actuator cap experienced by the min-time policy is not in place for the LQR policy. If it is the LQR policy will never do as well as the min-time policy, and will not limit towards it either as once hard limits on actuation magnitude are in place LQR loses its optimality guarantees and can start performing very poorly.

The LQR solution required 9.89 seconds to reach the goal using $Q = .25I$. The min-time policy required 4.43 seconds to reach the goal. Using $Q = 100I$, however, resulted in a time of 4.04 seconds. This is due to the LQR policy not having bounds on actuation, and thus being able to outperform the min-time solution through the use of much larger actuator magnitudes than the min-time was permitted.

MIT OpenCourseWare
<http://ocw.mit.edu>

6.832 Underactuated Robotics
Spring 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.