6.849	Lecture 16	Nov. 8, 2010

Gluing results: [Demaine, Demaine, Lubiw, O'Rourke 2002]

(algorithms & combinatorics)

	# gluings & enumeration alg.	decision algorithm
general	$2^{\Theta(n)}$	$2^{O(n)}$ *
edge-to-edge gluing	$2^{\Theta(n)}$	$n^{O(1)}$
bounded sharpness polygon	$n^{\Theta(1)}$	$n^{O(1)}$

lower bounds on worst-case # gluings (⇒ output size)

* $\boxed{\text{OPEN}}$: general polynomial-time decision algorithm (does the polygon have *any* gluing?)

Combinatorial type of gluing =

1. abstract gluing tree (without lengths)
2. specification of which polygon vertices & edges come together at each gluing-tree junction (including leaves & all points with ≥ 1 polygon vertex)

— count just combinatorially distinct gluings
⇒ remove infinities

Edge-to-edge gluing = only vertices at tree junctions
= only glue whole polygon edges to whole polygon edges

Bounded-sharpness polygon = all angles $\le 360° - \varepsilon$, for some $\varepsilon > 0$ indep. of n
(e.g., convex polygons)

$2^{O(n)}$ upper bound on gluings: (combinatorially distinct)

- each leaf in gluing tree is a vertex or fold point

$\Rightarrow \leq n+4$ leaves

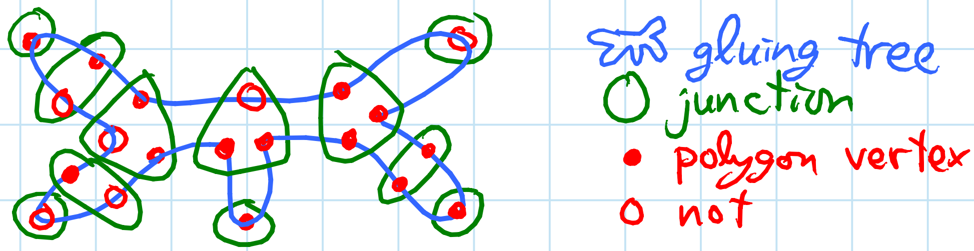
$\Rightarrow O(n)$ tree junctions of degree $\neq 2$

- $\leq n$ tree junctions with ≥ 1 polygon vertex (which includes all degree-2 junctions)

$\Rightarrow O(n)$ tree junctions

$\Rightarrow 2^{O(n)}$ possible abstract trees

- $2^{O(n)}$ choices of where each tree junction has polygon vertices (yes/no for each)

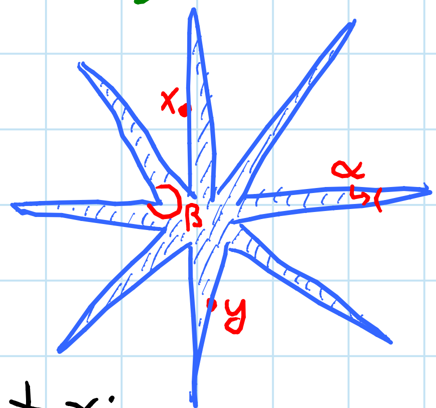


- n choices for which is first polygon vertex

$\Rightarrow 2^{O(n)} \cdot 2^{O(n)} \cdot n = 2^{O(n)}$ comb. distinct gluings

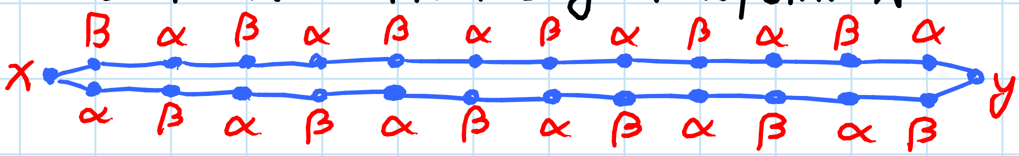
$2^{\Omega(n)}$ lower bound on gluings: (even edge-to-edge)

- sharpening: $\alpha \to \varepsilon$
  $\& \ \beta \to 360^\circ - \frac{360^\circ}{n/2} - \varepsilon'$
  $\ll 360^\circ - n\varepsilon$

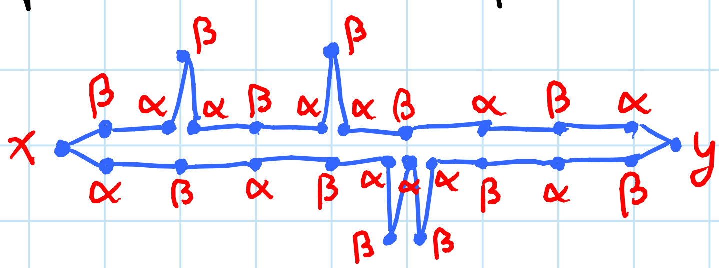


$\Rightarrow$ can glue any number of α's into a β (but not >1 β)

- perimeter halve from edge midpoint x:



- zip $n/4$ β's on top & $n/4$ β's on bottom:



- always an Alexandrov gluing

$\approx 2^{n/4} \cdot 2^{n/4} = 2^{n/2}$ gluings

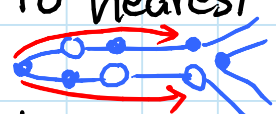
$n^{O(1)}$ upper bound for bounded sharpness:

$\Theta(1)$, not $O(1/n)$

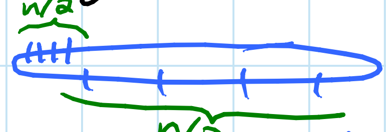
- suppose every vertex has angle $\le 360° - \varepsilon$
- $\Rightarrow$ every leaf in gluing tree has curvature $\ge \varepsilon$ (assuming $\varepsilon \le 180°$ to allow fold points)
- $720°$ total curvature
- $\Rightarrow L \le 720°/\varepsilon = O(1/\varepsilon)$ leaves
- e.g. $L \le 4$ if polygon convex

$n^{O(L)}$ upper bound for L leaves:

- $< L$ internal junctions of degree ≥ 3
- $2^{O(L)}$ abstract trees on these junctions + leaves
  i.e. on all but degree-2 junctions
- $\binom{O(n)}{O(L)}$ assignments of polygon vertices & edges to these tree junctions
- for vertex leaf, can measure distances to nearest branching junction (which has a vertex)



  $\Rightarrow$ determine this part of gluing; induct
- trouble: fold point (edge) at a leaf
  - especially with rolling belt
  - $O(n^2)$ combinatorial types in rolling belt
    ($O(n^2)$ events during rolling)



  - $n^{O(1)}$ over ≤ 3 belts

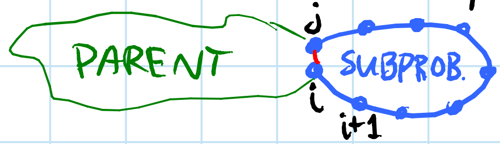
tight for $L=2$

- $\Rightarrow \binom{O(n)}{O(L)} \le n^{O(L)}$ & $2^{O(n)}$ total
- in fact: $O(n^{2L-2})$, $O(n^4)$ for $L=4$, $O(n^2)$ for $L \le 3$

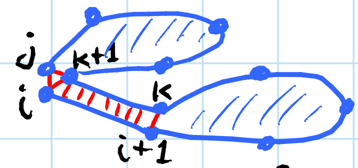
$\boxed{\text{OPEN}}$: better bounds for $L > 2$? $2^{O(L)} n^{O(1)}$?

Dynamic program for edge-to-edge gluing: [Lubiw & O'Rourke 1996]

- subproblem = chain $v_i \cdots v_j$ of the polygon whose endpoints are glued together



- to which edge does (ccw) first edge $v_i v_{i+1}$ of chain glue?
- at most n choices $v_k v_{k+1}$: (try them all)



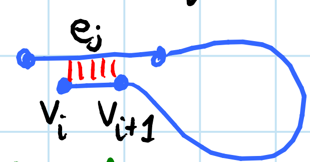
- get two smaller subproblems: $v_{i+1} \cdots v_k$ & $v_{k+1} \cdots v_j$
- to list all foldings:
  - combine all pairs of solutions to subproblems
  - discard any with negative curvature
  - exponential, but optimal in worst case
- decision problem in polynomial time:
  - each subproblem just reports one gluing, with min. possible total angle glued at v_i & v_j
    - take min over all (Alex.) combined options
    - ∞ if no Alexandrov gluing
  - if any gluing allows parent to be Alexandrov, this will be one
  - $O(n^2)$ subproblems, $O(n)$ time each

  $\Rightarrow O(n^3)$ time

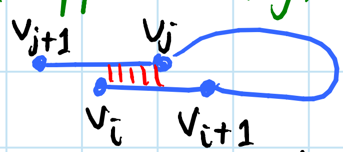
Dynamic program for general gluing: [Lubiw 2000; Hirata 2000]

- two types of subproblems:
  - chain $v_i \cdots v_j$ with v_i glued to v_j (v_j, v_{j+1})
  - chain $v_i \cdots e_j$ with v_i glued to point along e_j
  - (– chain $e_i \cdots v_j$ symmetrically)
- answer to subproblem = list of options, each:
  - combinatorial type of gluing
  - total angle of material glued at v_i
  - in $v_i \cdots e_j$ case, interval $\subseteq (0, |e_j|)$ of points along e_j to which v_i can glue in this option
- to solve $v_i \cdots e_j$ (say), there are two cases:

① zip: glue interval after v_i to interval before e_j point
  - stop at first vertex, either:

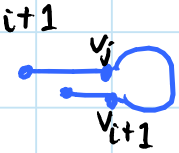


ⓐ $v_{i+1} \Rightarrow$ recurse on $v_{i+1} \cdots e_j$
  - v_i's interval = v_{i+1}'s, offset by $+|e_i|$, clipped at $|e_j|$

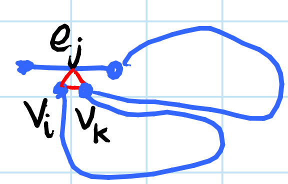


ⓑ $v_j \Rightarrow$ recurse on $e_i \cdots v_j$
  - v_i's interval = v_j's, cropped at $|e_j|$

ⓒ both v_{i+1} & $v_j \Rightarrow$ recurse on $v_{i+1} \cdots v_j$



② tug: glue another vertex v_k against v_i
  - assume v_k is smallest such
  - $\Rightarrow$ recurse on $v_i \cdots v_k$ with initial zip
  - recurse on $v_k \cdots e_j$ (zip or tug) – v_i's interval = v_k's

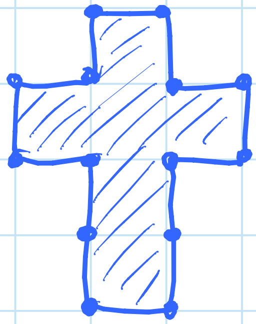


- in $v_i \cdots v_j$ subproblem, tug can glue edge e_k against v_i & v_j
- time = $O(n^3) \cdot$ # gluings at any stage

$$= \begin{cases} 2^{O(n)} & \text{in general} \\ n^{O(1)} & \text{for bounded sharpness} \end{cases}$$

Case studies:

Latin cross:

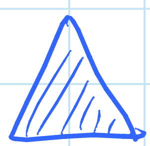


[Demaine, Demaine, Lubiw, O'Rourke]

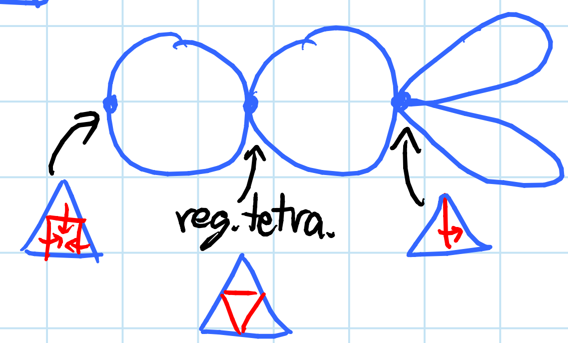
- 5 edge-to-edge gluings
  [Metamorphosis of the Cube]
- 85 general gluings $= 42 \cdot 2 + 1$ → symmetric
- 23 different polyhedra:
  - cube
  - 2 flat quads
  - 7 tetrahedra
  - 3 pentahedra
  - 4 hexahedra
  - 6 octahedra

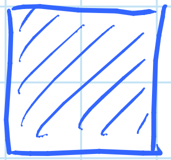
Hirata's Half-Length Theorem: [Hirata 2000]
if edge lengths are integers & no rolling belts
then suffices to cut each edge at midpoint
& use edge-to-edge gluings

Equilateral triangle:



- 2 flat polyhedra
- 4 continua:
  - hexahedra
  - pentahedra
  - tetrahedra



Square:  [Alexander, Dyson, O'Rourke 2003]

- continuua of 5 combinatorial types:
  - tetrahedra
  - 2 pentahedra
  - hexahedra
  - octahedra
- 4 flat polyhedra:
  - right
  - square
  - rect.
  - pentagon