Recall:

	edge unfolding	general unfolding
convex polyhedra	OPEN	ALWAYS
nonconvex polyhedra	NOT ALWAYS	OPEN

Vertex unfolding: [Demaine, Eppstein, Erickson, Hart, O'Rourke 2003]
 — different relaxation of edge unfolding
 — still cut only along polyhedron edges (all!)
 — require connectivity just through vertices (vs. edges)
 (like hinged dissection)

Example:
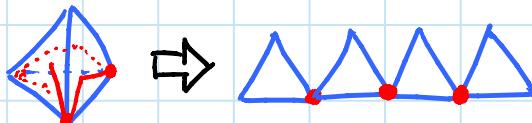
Universality: every connected triangulated manifold
 has a vertex unfolding, computable in linear time

① construct facet path going from facet
 to facet along vertex adjacencies,
 visiting every facet exactly once,
 not repeating a vertex twice in a row
 [next page; also Bartholdi & Goldsman 2004]
 ↳ but O(n²) time

② unfold each facet into vertical slab:

⇒ no overlap
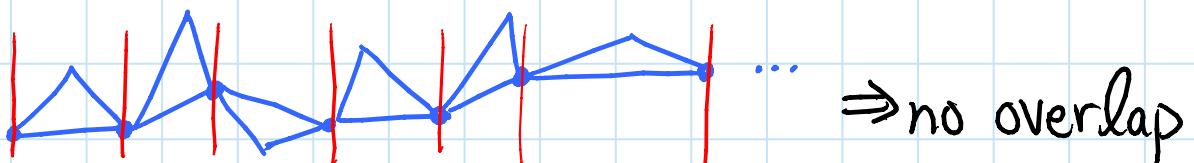
Vertex unfolding: (cont'd)

## Constructing a facet path for 2D surfaces:

① Cut edges until facets are connected in tree-like fashion
   - only removes connections
     $\Rightarrow$ only harder to find path
   - duplicates vertices but can repeat
   $\Rightarrow$ triangulated "polygon"
     (may self-overlap)
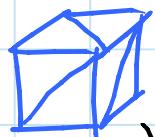
② Color "ears" = $\triangle$s with one adjacent triangle & two boundary edges
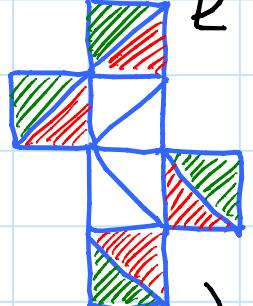
③ Color ears in what remains
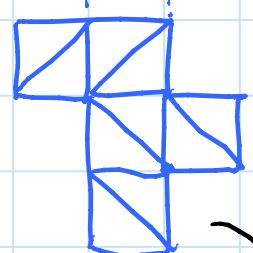
④ Remove second ear & 1 or 2 first ear(s):
   - either [diagram] or [diagram]
     "dunce cap"   "Mickey Mouse"
   - recurse on remainder
   - put back with [diagram] or [diagram]
   - base case: nothing or [triangle] or [diagram]
   $\Rightarrow$ every vertex but 2 from base case have even number of connections
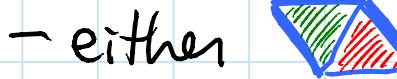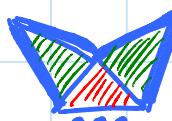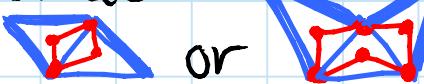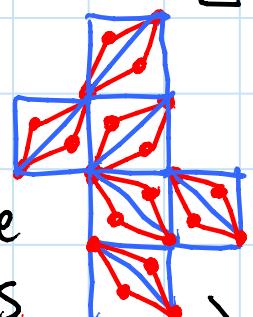
⑤ Connect components by local switches:
   [diagram] $\Rightarrow$ [diagram]
                          same degrees
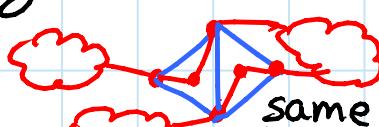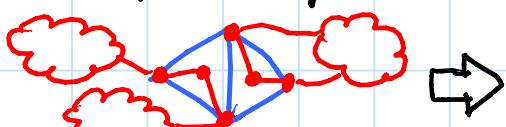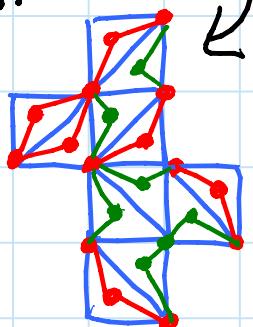
⑥ (Noncrossing) Eulerian path
   bonus: get cycle $\Leftrightarrow$ not "checkered":
        2-color $\triangle$s, only red $\triangle$s on boundary

2

Vertex unfolding: (cont'd)

General vertex unfolding is trivial: triangulate

OPEN: does every convex polyhedron have a vertex unfolding?
- facet path $\not\Rightarrow$ vertex unfolding:

  no slabs repeat 3 times

- facet path may not exist:
  - truncated cube has 8 disjoint △s & 6 ⬡s
  - not enough ⬡s to put between △s
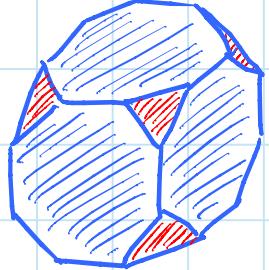
OPEN: does every polyhedron with holeless faces have a vertex unfolding?

cf.:
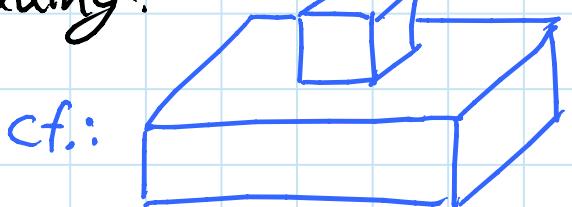
Orthogonal polyhedra: (all faces perpendicular to coord. axis)
generally unfoldable if genus Ø

[Damian, Flatland, O'Rourke 2007]

Proof: slice through every vertex with y-plane (xz)
- — y-faces encompassed = side faces
- — x & z-faces (yz & xy) = band faces
- — band faces between two y-planes form a cycle = band
- — bands form a tree structure, connecting parent to child with thin side faces
- — idea: visit bands in roughly depth-first order
  - — unfold to proceed always rightward
  - — side faces just attach above/below
- — unfold leaf band with spiral path:
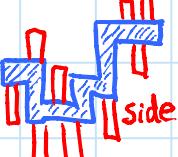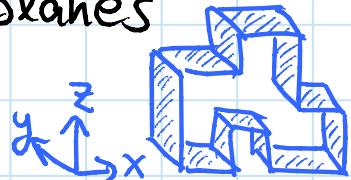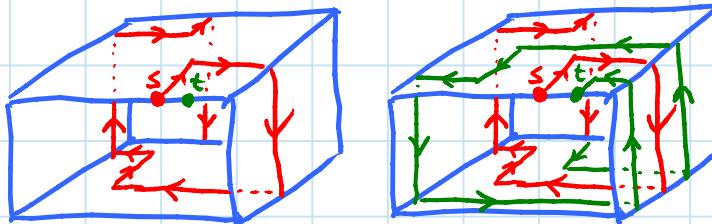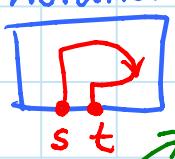
notation:

fewer turns (4 vs. 6) possible [Cheung 2007]

(thicken)

unfold ⇒

- — visit −y children with alternating path:

y ↑
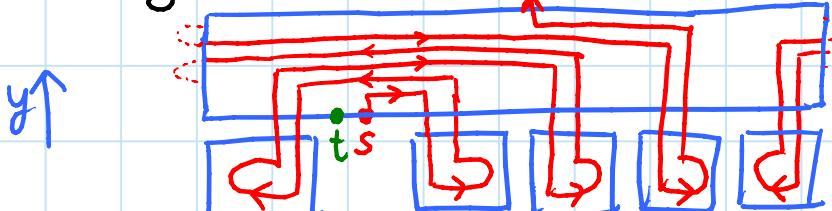
- — similarly visit +y children
- — double-back along entire subtree to return to parent @ t

4
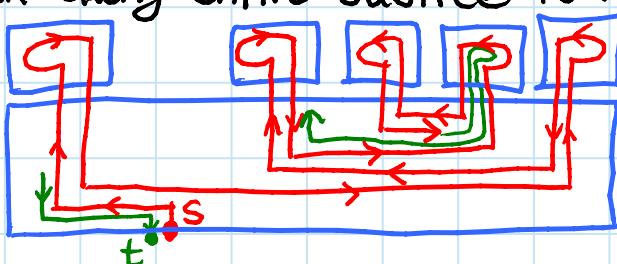
Orthogonal polyhedra: (cont'd)
Grid = slice through every vertex with x, y, z-planes
Grid unfolding = just cut along grid

OPEN : does every orthogonal polyhedron
        have a grid unfolding?
        — analog of edge unfolding
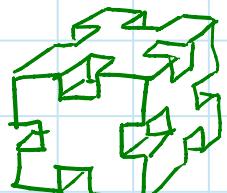
edge unfolding impossible

Refinement = divide each grid rectangle into $k \times k$

goal: minimize k

Summary:

type	$k \times k$	ref
— general	$2^{O(n)} \times 2^{O(n)}$	[DFO 2007] (prev. page)
— general vertex-unf.	1×1	[DFO 2006]
— Manhattan towers	5×4	[DFO 2005]

↳ connected y=0 base, y-plane slices shrink as y increases

|		
— orthoterrain (rect. base)	1×1	[O'Rourke 2007]
— orthostacks	2×1	[Biedl et al. 1998]

↳ every y-plane slice is connected

— orthostacks vertex-unf.	1×1	[Demaine, Iacono, Langerman 2006]
— orthoconvex orthostacks	1×1	[Damian & Meijer 2004]

↳ y-plane slices are orthogonally convex: x & y slices connected

— orthotubes	1×1	[Biedl et al. 1998]

↳ unit cubes connected face-to-face in open/closed chain

— well-separated orthotrees	1×1	[DFMeijerO 2005]

↳ unit cubes connected face-to-face in a tree
↳ no two branching cubes are adjacent

Orthogonal polyhedra: (cont'd)

OPEN: $n^{O(1)} \times n^{O(1)}$ refinement in general?

OPEN: $\Omega(n) \times \Omega(n)$ lower bound?

OPEN: power of $O(1) \times O(1)$ refinement?
  — e.g. orthotrees

OPEN: power of 1×1 refinement?
  — e.g. orthostacks, Manhattan towers

OPEN: orthogonal polyhedra of higher genus?

OPEN: general unfolding of arbitrary polyhedra of genus $\emptyset$?

PROJECT: implement $2^{O(n)} \times 2^{O(n)}$ method for unfolding orthogonal polyhedra of genus $\emptyset$

Cauchy's Rigidity Theorem: [Cauchy 1813; Steinitz 1934]

if two convex polyhedra are combinatorially equivalent
same graph/incidence structure

& corresp. faces are congruent, then polyhedra are congruent

Proof by contradiction: consider counterexample P, P'
- for each vertex pair v, v':
    slice polyhedron with ε-sphere at v, v'
    $\Rightarrow$ spherical polygon, edge lengths = face angles,
    angles = dihedral angles
- faces congruent $\Rightarrow$ edge lengths match in v vs. v'
- P, P' incongruent $\Rightarrow$ angles don't match for <u>some</u> v, v'
- label vertex in v's spherical polygon + if larger angle in v,
    $\emptyset$ if equal angles, and − if smaller angle in v
- can't be all + (& $\emptyset$) or all − (& $\emptyset$) by:
    <u>Cauchy Arm Lemma</u>: opening all angles of a
    convex open chain increases distance of endpts.
    (in plane & sphere)
 $\Rightarrow$ edge length would not be preserved
$\Rightarrow$ ≥2 alternations
- can't be just 2 by Cauchy Arm Lemma:
$\Rightarrow$ ≥4 alternations
$\Rightarrow$ # alternations ≥ 4V in subgraph of +/− edges
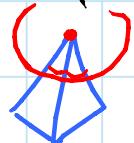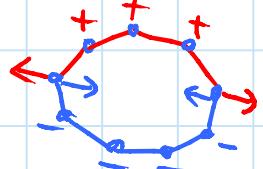
<div align="center">7</div>

Proof of Cauchy's Rigidity Theorem: (cont'd)

- $\pm$ labels extend to entire edge of polyhedron
- $\pm$ alternation at vertex corresponds one-to-one
  $\pm$ alternation in incident face
- $\leq 2k$ alternations in face of $2k$ or $2k+1$ edges
- $\Rightarrow 4V \leq$ #alternations $\leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 7f_7 + \cdots$
- $E = \frac{1}{2}(3f_3 + 4f_4 + 5f_5 + \cdots)$ (handshaking)
- $V = 2 + E - F$ Euler's Theorem
  $= 2 + \frac{1}{2}(f_3 + 2f_4 + 3f_5 + \cdots)$
- $\Rightarrow 4V = 8 + 2f_3 + 4f_4 + 6f_5 + \cdots \longrightarrow$ contradiction $(+8)$
  $\square$

Uniqueness of folding: [Alexandrov 1941]

- suppose you glue polygon's boundary to itself
- what convex polyhedra can it form?
- every edge will be a shortest path
- draw all shortest paths between vertices
  (points of nonzero curvature)
- $\Rightarrow$ fix combinatorial & facial structure
- Cauchy's Rigidity Theorem $\Rightarrow \leq 1$ convex realization

NEXT LECTURE: when is there ≥ 1 convex realization?

6.849 Geometric Folding Algorithms: Linkages, Origami, Polyhedra
Fall 2012